A Minkowski-*like* Inequality over $\mathbb{R}^n$

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Abstract

The discrete case of the Minkowski inequality for *p* =2 is a well-known triangle inequality over the set of complex. This paper presents a new Minkowski-*like* inequality over the set of reals.

Introduction

The triangle inequality states that given any triangle with sides of length, *a, b,* and *c*, then $c < a + b$. Equivalently, for complex numbers z_1 and z_2 we write $|z_1 + z_2| \leq |z_1| + |z_2|$, where $|z|$ represents the norm of the vector. If we define

$$|a| = \left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}}$$

to be the norm of vector $a = (a_1,\ a_2, \cdots, a_n) \in \mathbb{C}^n$ where $\mathbb{C}^n$ is the usual *n* dimension vector space over the reals, we have a triangle inequality in $\mathbb{C}^n$. The discrete case of Minkowski's Inequality for *p* = 2 is such a triangle inequality. Here is a popular proof.

Theorem. Let $a_1,\ a_2, \cdots, a_n$ and $b_1,\ b_2, \cdots, b_n$ be complex numbers, then

$$\left(\sum_{j=1}^{n} \left|a_j + b_j\right|^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{n} \left|a_j\right|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} \left|b_j\right|^2\right)^{\frac{1}{2}}.$$

Proof [1, p.25]. Let

$$A = \sum_{j=1}^{n} \left|a_j\right|^2, \quad B = \sum_{j=1}^{n} \left|b_j\right|^2, \text{and } C = \sum_{j=1}^{n} a_j \overline{b_j}.$$

If *B* = 0, then $b_k = 0\ \forall\ k$ and the conclusion is trivial. If $B > 0$, then

$$\sum_{j=1}^{n} \left|a_j + b_j\right|^2 = \sum_{j=1}^{n} \left(a_j + b_j\right)\left(\overline{a_j + b_j}\right)$$

$$= \sum_{j=1}^{n} a_j \overline{a_j} + \sum_{j=1}^{n} a_j \overline{b_j} + \sum_{j=1}^{n} b_j \overline{a_j} + \sum_{j=1}^{n} b_j \overline{b_j}$$

$$= A + C + \bar{C} + B$$

$$= A + 2Re(C) + B$$

$$\leq A + 2|C| + B$$

$$\leq A + 2\sqrt{A}\sqrt{B} + B$$

$$= \left(\sqrt{A} + \sqrt{B}\right)^2.$$

The second inequality is Cauchy's inequality (or CBS-Inequality).

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If we define

$$|a| = \left(\sum_{k=1}^{n} a_k{}^q\right)^{\frac{1}{p}}, \qquad p = 2, 3, \dots, q \in \mathbb{N}, p > q$$

for vector $a = (a_1,\ a_2, \cdots, a_n) \in \mathbb{R}^n, a_k \geq 0$, then we have a Minkowski-*like* inequality over $\mathbb{R}^n$.

Theorem. Let $a_1,\ a_2, \cdots, a_n$ and $b_1,\ b_2, \cdots, b_n$ be real values where $a_{k,}\ b_k \geq 0\ \forall\ k$, then for p = 2, 3, … , $q \in \mathbb{N}$, $p > q$,

$$\left(\sum_{j=1}^{n} \left(a_j + b_j\right)^q\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{n} a_j{}^q\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} b_j{}^q\right)^{\frac{1}{p}}.$$

Proof.
If $a_k + b_k = 0\ \ \forall\ k$, then the conclusion is trivial. Using induction set p = 2 and we have,

$$\left(\sum_{j=1}^{n} \left(a_j + b_j\right)^q\right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{n} a_j{}^q\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n} b_j{}^q\right)^{\frac{1}{2}}.$$

This is a case of the Minkowski's Inequality over the set of reals. Now, assume

$$\left(\sum_{j=1}^{n} \left(a_j + b_j\right)^q\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^{n} a_j{}^q\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{n} b_j{}^q\right)^{\frac{1}{p}} \text{ for } p > 2 \text{ then}$$

$$\left(\sum_{j=1}^{n}(a_j+b_j)^q\right)^{\frac{1}{p+1}}$$

$$=\left(\sum_{j=1}^{n}(a_j+b_j)^q\right)^{\frac{1}{p}}\left(\sum_{j=1}^{n}(a_j+b_j)^q\right)^{\frac{-1}{p(p+1)}}$$

$$\leq\left[\left(\sum_{j=1}^{n}a_j{}^q\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}b_j{}^q\right)^{\frac{1}{p}}\right]\left(\sum_{j=1}^{n}(a_j+b_j)^q\right)^{\frac{-1}{p(p+1)}}$$

$$=\frac{\left(\sum_{j=1}^{n}a_j{}^q\right)^{\frac{1}{p}}}{\left(\sum_{j=1}^{n}(a_j+b_j)^q\right)^{\frac{1}{p(p+1)}}}+\frac{\left(\sum_{j=1}^{n}b_j{}^q\right)^{\frac{1}{p}}}{\left(\sum_{j=1}^{n}(a_j+b_j)^q\right)^{\frac{1}{p(p+1)}}}$$

$$\leq\frac{\left(\sum_{j=1}^{n}a_j{}^q\right)^{\frac{1}{p}}}{\left(\sum_{j=1}^{n}a_j{}^q\right)^{\frac{1}{p(p+1)}}}+\frac{\left(\sum_{j=1}^{n}b_j{}^q\right)^{\frac{1}{p}}}{\left(\sum_{j=1}^{n}b_j{}^q\right)^{\frac{1}{p(p+1)}}}$$

$$=\left(\sum_{j=1}^{n}a_j{}^q\right)^{\frac{1}{p+1}}+\left(\sum_{j=1}^{n}b_j{}^q\right)^{\frac{1}{p+1}}.$$

Hence the inequality,

$$\left(\sum_{j=1}^{n}(a_j+b_j)^q\right)^{\frac{1}{p}}\leq\left(\sum_{j=1}^{n}a_j{}^q\right)^{\frac{1}{p}}+\left(\sum_{j=1}^{n}b_j{}^q\right)^{\frac{1}{p}}$$

is true for $p=2,3,\dots,q\in\mathbb{N},p>q$.

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Reference.

1. K. R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth, Inc., 1981.