The Fundamental Property of Nagel Point – A New Proof

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Abstract*:* *In this article, we study the new proof of two fundamental properties of Nagel Point.*

Keywords: *Medial triangle, Incenter, Extouch Points, Splitters.*

1. INTRODUCTION

Given a triangle ABC, let T_A, T_B and T_C be the extouch points in which the A-excircle meets line BC, the B-excircle meets line CA, and C-excircle meets line AB respectively. The lines AT_A, BT_B, CT_C concur in the Nagel point $\mathbf{N_G}$ of triangle ABC. The Nagel point is named after Christian Heinrich von Nagel, a nineteenth-century German mathematician, who wrote about it in 1836. The Nagel point is sometimes also called the bisected perimeter point, and the segments AT_A, BT_B, CT_C are called the triangle's splitters. (See figure-1)**[6].**

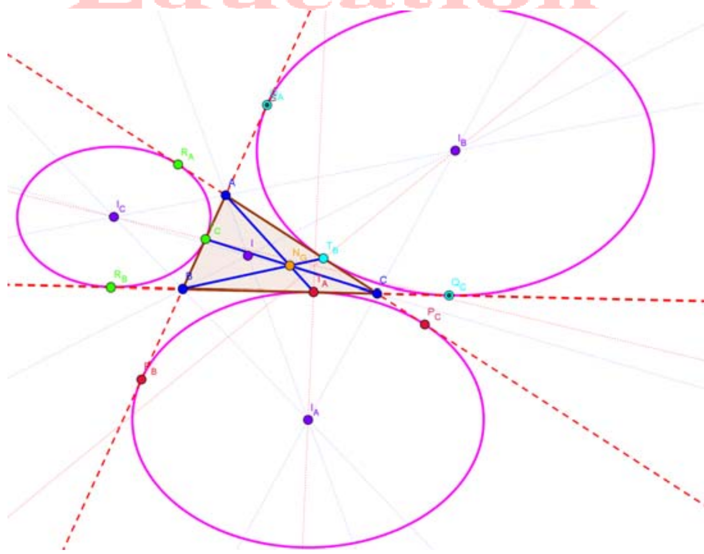

Figure-1 Nagel's Point (N_G)

In this short note we study a new proof of the fundamental property of this point, it is stated as "*The Nagel point of Medial Triangle acts as Incenter of the reference triangle*" (see figure-2), the synthetic proof of this property can be found in [8] . In this article we give a probably new and shortest proof which is purely based on the metric relation of Nagel's Point.

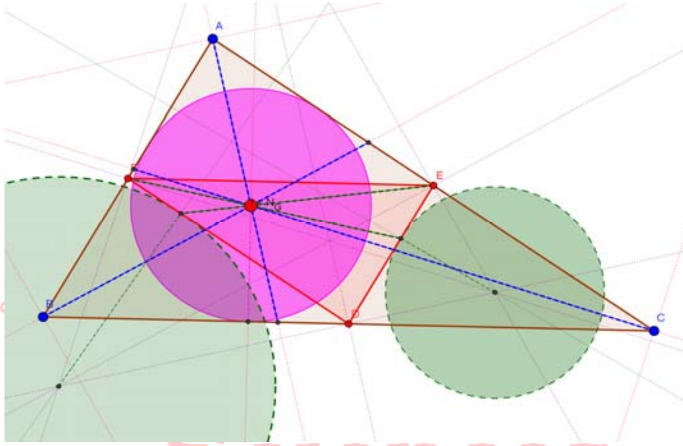

Figure-2, The Nagel's Point of ΔDEF is acts as Incenter of ΔABC

2. Notation and background

Let ABC be a non equilateral triangle. We denote its side-lengths by a, b, c, perimeter by 2s, its area by Δ and its circumradius by R, its inradius by r and exradii by r_1, r_2, r_3 respectively. Let T_A, P_B and P_C be the extouch points in which the *A*-excircle meets the sides *BC, AB and AC*, let T_B, Q_A and Q_C be the extouch points in which the *B*-excircle meets the sides *AC, BA and BC* , let T_C, R_A and R_B be the extouch points in which the *C*-excircle meets the sides *AB, CA and CB*.

The Medial Triangle:

The triangle formed by the feet of the medians is called as Medial triangle. Its sides are parallel to the sides of given triangleABC. By Thales theorem the sides, semi perimeter and angles of medial triangle are $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}, \frac{s}{2}$, A, B and C respectively. Its area is $\frac{\Delta}{4}$, circumradius $\frac{R}{2}$, inradius $\frac{r}{2}$.(refer [1], **[2], [3]**and**[4])**. *Before proving our main task let us prove some prepositions related to Nagel point.*

3. Prepositions

Preposition: 1

If AT_A, BT_B, CT_C are the splitters then $BT_A = s - c = AT_B$,
$CT_A = s - b = AT_C$ *and* $CT_B = s - a = BT_C$

Proof:
We are familiar with the fact that "*From an external point we can draw two tangents to a circle whose lengths are equal*".

....(Ω)

So $BP_B = BT_A = x$ (let) and $CP_C = CT_A = y$ (let), it is clear that $a = BC = BT_A + T_AC = x+y$ ....(1.1)
In the similar manner using (Ω), we have $AP_B = AP_C$, it implies $c+x = b+y$,
It gives $b\text{-}c = x\text{-}y$

….(1.2)

By solving (1.1) and (1.2), we can prove $x = s - c$ and $y = s - b$
That is $BT_A = s - c$ and $CT_A = s - b$
Similarly we can prove $CT_B = s - a = BT_C$, $AT_B = s - c$, $AT_C = s - b$

Preposition: 2

If AT_A, BT_B, CT_C are the splitters of the triangle ABC then they are concurrent and the point of concurrence is the Nagel Point N_G of the triangle ABC.

Proof:
By Preposition :1, we have $BT_A = s - c = AT_B$,
$CT_A = s - b = AT_C$ and $CT_B = s - a = BT_C$

Clearly $\frac{AT_C}{T_CB} \cdot \frac{BT_A}{T_AC} \cdot \frac{CT_B}{T_BA} = \frac{s-b}{s-a} \cdot \frac{s-c}{s-b} \cdot \frac{s-a}{s-c} = 1$

Hence by the converse of Ceva's Theorem, the three splitters AT_A, BT_B, CT_C are concurrent and the point of concurrency is called as Nagel's Point N_G .

Preposition: 3

If AT_A, BT_B, CT_C are the splitters of the triangle ABC then the length of each splitter is given by $AT_A = \sqrt{s^2 - \frac{4\Delta^2}{a(s-a)}}$,

$BT_B = \sqrt{s^2 - \frac{4\Delta^2}{b(s-b)}}$ *and* $CT_C = \sqrt{s^2 - \frac{4\Delta^2}{c(s-c)}}$

Proof:
Clearly for the triangle ABC, the line AT_A is a cevian,
Hence by Stewarts theorem we have

$$AT_A^{\,2} = \frac{BT_A.AC^2}{BC} + \frac{CT_A.AB^2}{BC} - BT_A.CT_A$$

It implies $AT_A^{\,2} = \frac{(s-c)b^2}{a} + \frac{(s-c)c^2}{a} - (s-b)(s-c)$

Further simplification gives $AT_A = \sqrt{s^2 - \frac{4\Delta^2}{a(s-a)}}$

Similarly we can prove $BT_B = \sqrt{s^2 - \frac{4\Delta^2}{b(s-b)}}$ and $CT_C = \sqrt{s^2 - \frac{4\Delta^2}{c(s-c)}}$

Preposition: 4

The Nagel Point N_G of the triangle ABC divides each splitters in the ratio given by

$AN_G : N_GT_A = a : s-a$, $BN_G : N_GT_B = b : s-b$ *and*

$CN_G : N_GT_C = c : s-c$

Proof:

We have by preposition :1, $BT_A = s-c$ and $CT_A = s-b$

Now for the triangle ABT_A , The line T_CN_GC acts as transversal,

So *Menelaus Theorem* we have $\frac{AT_C}{T_CB} . \frac{BC}{CT_A} . \frac{T_AN_G}{N_GA} = 1$

It implies $AN_G : N_GT_A = a : s-a$

Similarly we can prove $BN_G : N_GT_B = b : s-b$ and

$CN_G : N_GT_C = c : s-c$

Preposition: 5

If D, E, F are the foot of medians of Δ ABC drawn from the vertices A,B,C on the sides BC, CA, AB and M be any point in the plane of the triangle then

$\boldsymbol{4DM^2 = 2CM^2 + 2BM^2 - a^2}$ $\boldsymbol{4EM^2 = 2CM^2 + 2AM^2 - b^2}$ *and*

$\boldsymbol{4FM^2 = 2AM^2 + 2BM^2 - c^2}$

Proof:

The proof of above Preposition can be found in [2] and [3]

Preposition : 6

If a, b, c are the sides of the triangle ABC, and if s,R,r and Δ are semi perimeter, Circumradius, Inradius and area of the triangle ABC respectively then

1. $abc = 4R\Delta = 4Rrs$
2. $ab + bc + ca = r^2 + s^2 + 4Rr$
3. $a^2 + b^2 + c^2 = 2\left(s^2 - r^2 - 4Rr\right)$
4. $a^3 + b^3 + c^3 = 2s\left(s^2 - 3r^2 - 6Rr\right)$

Proof:

The Proof of above Preposition can be found in [3].

4. Main Results

Metric Relation of Nagel Point

Theorem 1
Let M be any point in the plane of the triangle ABC and if N_G is the Nagel Point of the triangle ABC then

$$N_G M^2 = \left(\frac{s-a}{s}\right)AM^2 + \left(\frac{s-b}{s}\right)BM^2 + \left(\frac{s-c}{s}\right)CM^2 + 4r^2 - 4Rr$$

Proof:

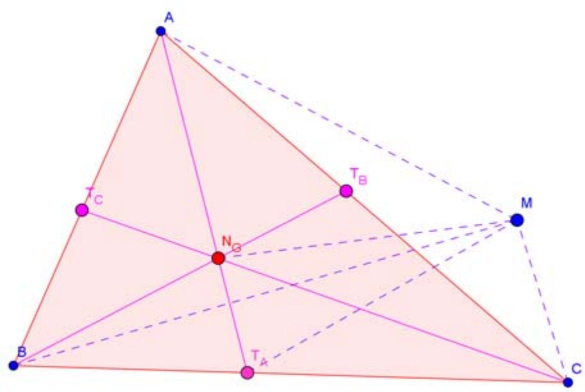

Figure-3

Let ''M'' be any point of in the plane of ΔABC,
Since $T_A M$ is a cevian for the triangle BMC,
Hence by applying *Stewart's theorem* for ΔBMC,

We get $T_A M^2 = \dfrac{BT_A . CM^2}{BC} + \dfrac{CT_A . BM^2}{BC} - BT_A . CT_A$

$$= \frac{(s-c)CM^2}{a} + \frac{(s-b)BM^2}{a} - (s-b)(s-c)$$

….(π)

Now for the triangle AMT_A, the line N_GM is a cevian,
So again by *Stewart's theorem*,

We have $N_GM^2 = \frac{AN_G.T_AM^2}{AT_A} + \frac{N_GT_A.AM^2}{AT_A}$ - $AN_G.N_GT_A$

….(£)

By replacing T_AM, AN_G, N_GT_A Using (π), Preposition :3 and 4,
(£) can be rewritten as

$$N_GM^2 = \left(\frac{s-a}{s}\right)AM^2 + \left(\frac{s-b}{s}\right)BM^2 + \left(\frac{s-c}{s}\right)CM^2 - \frac{a}{s}(s-b)(s-c) - \frac{a(s-a)}{s^2}\left(s^2 - \frac{4\Delta^2}{a(s-a)}\right)$$

Further simplification gives

$$N_GM^2 = \left(\frac{s-a}{s}\right)AM^2 + \left(\frac{s-b}{s}\right)BM^2 + \left(\frac{s-c}{s}\right)CM^2 + 4r^2 - 4Rr$$

Theorem 2
If N'_G be the Nagel Point of medial triangle ΔDEF of triangle ABC and let M be any point in the plane of the triangle then

$$N'_GM^2 = \left(\frac{s'-a'}{s'}\right)DM^2 + \left(\frac{s'-b'}{s'}\right)EM^2 + \left(\frac{s'-c'}{s'}\right)FM^2 + 4(r')^2 - 4R'r'$$

where a', b', c', s', R', r' are corresponding sides, semi perimeter, circumradius, inradius of the medial triangle DEF.

Proof:
Replace N_G as N'_G and a, b, c, s, R, r as a', b', c', s', R', r' and the vertices A, B, C as D, E, F in Theorem -1 we get Theorem -2

Theorem 3
If I is the Incenter of the triangle ABC whose sides are a, b and c and M be any point in the plane of the triangle then

$$IM^2 = \frac{a\,AM^2 + b\,BM^2 + c\,CM^2 - abc}{a+b+c}$$

Proof:
The proof above Theorem can be found in [3] and [1].

The First Fundamental Property of Nagel Point

If N'_G be the Nagel Point of medial triangle ΔDEF of triangle ABC , I is the Incenter of triangle ABC and let M be any point in the plane of the triangle then $N'_G M = IM$

That is the Nagel point of Medial Triangle acts as Incenter of the reference triangle

Proof:
Using Theorem 2
We have

$$N'_G M^2 = \left(\frac{s'-a'}{s'}\right) DM^2 + \left(\frac{s'-b'}{s'}\right) EM^2 + \left(\frac{s'-c'}{s'}\right) FM^2 + 4\left(r'\right)^2 - 4R'r'$$

Using the properties of medial triangle Replace a', b', c', s', R', r' with $\frac{a}{2}, \frac{b}{2}, \frac{c}{2}, \frac{s}{2}, \frac{R}{2}, \frac{r}{2}$

And By replacing DM, EM, FM using Preposition-5, we get

$$N'_G M^2 = \left(\frac{s-a}{4s}\right)\left(2BM^2 + 2CM^2 - a^2\right) + \left(\frac{s-b}{4s}\right)\left(2AM^2 + 2CM^2 - b^2\right) + \left(\frac{s-c}{4s}\right)\left(2BM^2 + 2AM^2 - c^2\right) + \left(r\right)^2 - Rr$$

It implies

$$N'_G M^2 = \frac{1}{4s}\left[2aAM^2 + 2bBM^2 + 2cCM^2 - a^2(s-a) - b^2(s-b) - c^2(s-c)\right] + \left(r\right)^2 - Rr$$

Using Theorem 3, we have

$$2aAM^2 + 2bBM^2 + 2cCM^2 = 4sIM^2 + 2abc = 4s(IM^2 + 2Rr)$$

And using Preposition :6, we have

$$a^2(s-a) + b^2(s-b) + c^2(s-c) = s(a^2 + b^2 + c^2) - (a^3 + b^3 + c^3)$$

$$= 2s\left(s^2 - r^2 - 4Rr\right) - 2s\left(s^2 - 3r^2 - 6Rr\right) = 4s(r^2 + Rr)$$

Hence $\boldsymbol{N_G M^2 = IM^2}$

That is the Nagel point of Medial Triangle acts as Incenter of the reference triangle. Further details about the Nagel Point refer [5], [6] and [8]

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