A converse of the mean value theorem for differentiable functions of one or more variables

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Abstract

Let $f : \Omega \rightarrow R$ be a continuously differentiable function of $\vec{x}$ in Ω, where $\Omega \subset R^N$ is an open, convex set and N ≥ 1. Let $\vec{c} \in \Omega$ be a given point. We prove that if there exists a vector $\vec{k}$ such that $\nabla f(\vec{c}) \bullet \vec{k}$ is not a local extremum value of $\nabla f(\vec{x}(t)) \bullet \vec{k}$ for points $\vec{x}(t) = \vec{c} + t\vec{k}$ on the line through the point $\vec{c}$ for $t \in I$, where I is an interval such that $\vec{x}(t) \in \Omega$ if $t \in I$, and if 0 is not an accumulation point of the set $S = \{t \in I \mid \nabla f(\vec{x}(t)) \bullet \vec{k} = \nabla f(\vec{c}) \bullet \vec{k}\}$, then there exist points $\vec{a}$, $\vec{b}$ in Ω such that $f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \bullet (\vec{b} - \vec{a})$.

Introduction

One version of the mean value theorem states that if $f : \Omega \rightarrow R$ is a continuously differentiable function of $\vec{x}$ on an open, convex set $\Omega \subset R^N$ and $\vec{a}$, $\vec{b}$ in Ω are given points, then there exists a point $\vec{c} \in \Omega$ such that $f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \bullet (\vec{b} - \vec{a})$ (see, e.g., [2]). The question to be considered here is: If $f : \Omega \rightarrow R$ is a continuously differentiable function of $\vec{x}$ on an open, convex set $\Omega \subset R^N$ and $\vec{c} \in \Omega$ is a given point, then do there exist points $\vec{a}$, $\vec{b}$ in Ω such that $f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \bullet (\vec{b} - \vec{a})$? In this paper, we prove that if there exists a vector $\vec{k}$ such that $\nabla f(\vec{c}) \bullet \vec{k}$ is not a local extremum value of $\nabla f(\vec{x}(t)) \bullet \vec{k}$ for points $\vec{x}(t) = \vec{c} + t\vec{k}$ on the line through the point $\vec{c}$ for $t \in I$, where I is an interval such that $\vec{x}(t) \in \Omega$ if $t \in I$, and if 0 is not an accumulation point of the set $S = \{t \in I \mid \nabla f(\vec{x}(t)) \bullet \vec{k} = \nabla f(\vec{c}) \bullet \vec{k}\}$, then there exist points $\vec{a}$, $\vec{b}$ in Ω such that $f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \bullet (\vec{b} - \vec{a})$.

Several authors have studied the converse of the mean value theorem for functions of one variable (see, e.g., Almeida [1], Mortici [3], Tong and Braza [4]). For example, Almeida [1] proved that if f is a continuous function on an interval $[a,b] \subset R$ and differentiable on the interval (a,b), then there exists an interval $(\alpha,\beta) \subset (a,b)$ with $c \in [\alpha,\beta]$ such that $f(\beta) - f(\alpha) = f'(c)(\beta-\alpha)$, if there exists $k_0 > 0$ such that $(c-k_0, c+k_0) \subset (a,b)$ and $f'(c-k) \le f'(c) \le f'(c+k)$ for all $k \in (0, k_0)$.

We have not seen any work by other researchers related to the converse of the mean value theorem for functions of several variables.

The paper is organized as follows: The main result, Theorem 1, is presented and proven in the next section. A lemma supporting the proof of Theorem 1 appears in the Appendix at the end of this paper.

A converse of the mean value theorem

The purpose of this paper is to prove the following theorem:

Theorem 1: *Let $f : \Omega \rightarrow R$ be a continuously differentiable function of $\vec{x} \in \Omega$, where $\Omega \subset R^N$ is an open, convex set and $N \ge 1$. Let $\vec{c} \in \Omega$ be a given point. If there exists a vector $\vec{k}$ such that $\nabla f(\vec{c}) \bullet \vec{k}$ is not a local extremum value of $\nabla f(\vec{x}(t)) \bullet \vec{k}$ for points $\vec{x}(t) = \vec{c} + t\vec{k}$ on the line through the point $\vec{c}$ for $t \in I$, where I is an interval such that $\vec{x}(t) \in \Omega$ if $t \in I$, and if 0 is not an accumulation point of the set $S = \{t \in I \mid \nabla f(\vec{x}(t)) \bullet \vec{k} = \nabla f(\vec{c}) \bullet \vec{k}\}$, then there exist points $\vec{a}$, $\vec{b}$ in Ω such that $f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \bullet (\vec{b} - \vec{a})$.*

Proof:

Suppose that there exists a vector $\vec{k}$ such that $\nabla f(\vec{c}) \bullet \vec{k}$ is not a local extremum value of $\nabla f(\vec{x}(t)) \bullet \vec{k}$ for points $\vec{x}(t) = \vec{c} + t\vec{k}$ on the line through the point $\vec{c}$ for $t \in I$, where I is an interval such that $\vec{x}(t) \in \Omega$ if $t \in I$, and suppose that 0 is not an accumulation point of the set

$S = \{t \in I \mid \nabla f(\vec{x}(t)) \bullet \vec{k} = \nabla f(\vec{c}) \bullet \vec{k}\}$.

Let $h(t) = \nabla f\ (\vec{c} + t\vec{k}) \bullet \vec{k}$. Then $h(0)$ is not a local extremum value of $h(t)$ for $t \in I$, and 0 is not an accumulation point of the set $S = \{t \in I \mid h(t) = h(0)\}$. It follows from Lemma 1 (which appears in the Appendix) that there exist numbers $t_1, t_2 \in I$ such that $t_1 < 0 < t_2$ and

$$h(0) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} h(\tau) d\tau .$$

Therefore, we obtain the following result:

$$h(0) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} h(\tau) d\tau$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \nabla f(\vec{c} + \tau\vec{k}) \bullet \vec{k} d\tau$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \nabla f(\vec{x}(\tau)) \bullet \vec{x}'(\tau) d\tau$$

$$= \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{df(\vec{x}(\tau))}{d\tau} d\tau$$

$$= \frac{1}{t_2 - t_1} (f(\vec{x}(t_2)) - f(\vec{x}(t_1)))$$

$$= \frac{1}{t_2 - t_1} (f(\vec{b}) - f(\vec{a}))$$

where we define $\vec{a} = \vec{x}(t_1) = \vec{c} + t_1 \vec{k}$ and we define $\vec{b} = \vec{x}(t_2) = \vec{c} + t_2 \vec{k}$. Note that $\vec{x}(t_1) \in \Omega$ and $\vec{x}(t_2) \in \Omega$ because $t_1 \in I$ and $t_2 \in I$ and because $\vec{x}(t)$ in Ω for $t \in I$.

Since $\vec{b} - \vec{a} = (t_2 - t_1)\vec{k}$, it follows that

$$h(0) = \nabla f(\vec{c}\ ) \bullet \vec{k} = \frac{1}{t_2 - t_1} \nabla f\ (\vec{c}\ ) \bullet (\vec{b} - \vec{a})$$

And since $h(0)=\dfrac{1}{t_2-t_1}(f(\vec{b})-f(\vec{a}))$, it follows that

$$f(\vec{b})-f(\vec{a})=\nabla f(\vec{c})\bullet(\vec{b}-\vec{a}).$$

This completes the proof of the theorem.

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References

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[3] C. Mortici, "A converse of the mean value theorem made easy", Internat. J. Math. Ed. Sci.Tech., **42** (2011), no. 1, 89--91.

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Appendix

Lemma 1: *Let $h: I\to R$ be a continuous function where $I \subset R$ is an open interval, and let $t_0\in I$ be a given number . If $h(t_0)$ is not a local extremum value of $h(t)$ in I , and t_0 is not an accumulation point of the set $S=\{t\in I \mid h(t)=h(t_0)\}$, then there exist numbers $t_1,t_2\in I$ such that*

$$t_1<t_0<t_2 \text{ and } h(t_0)=\frac{1}{t_2-t_1}\int_{t_1}^{t_2}h(\tau)d\tau.$$

Proof:

Suppose that $h(t_0)$ is not a local extremum value of $h(t)$ in I, and t_0 is not an accumulation point of the set $S=\{t\in I \mid h(t)=h(t_0)\}$. Since t_0 is not an accumulation point of the set $S=\{t\in I \mid h(t)=h(t_0)\}$, it follows that there exists a neighborhood $N\subset I$ of t_0 such that $h(t)\neq h(t_0)$ for all

$t \in N \setminus \{t_0\}$. Since h is continuous and $h(t_0)$ is not a local extremum value of h, it follows that either $h(t) > h(t_0)$ for $t > t_0$ and $h(t) < h(t_0)$ for $t < t_0$, or $h(t) < h(t_0)$ for $t > t_0$ and $h(t) > h(t_0)$ for $t < t_0$, where $t \in N$.

First, suppose that $h(t) > h(t_0)$ for $t > t_0$ and $h(t) < h(t_0)$ for $t < t_0$, where $t \in N$. Let $G(t) = \int_{t_0}^{t} h(\tau) - h(t_0) d\tau$. It follows that there exist numbers a_1, b_1 in N such that $a_1 < t_0 < b_1$ and $G(a_1) > 0$, $G(b_1) > 0$. Note that $G(t_0) = 0$. Therefore, by the continuity of $G(t)$ and the Intermediate Value Theorem, it follows that there exist numbers t_1, t_2 in N such that $G(t_1) = G(t_2)$ and $a_1 \le t_1 < t_0 < t_2 \le b_1$.
Then $\int_{t_0}^{t_1} h(\tau) - h(t_0) d\tau = \int_{t_0}^{t_2} h(\tau) - h(t_0) d\tau$, and it immediately follows that $h(t_0) = \dfrac{1}{t_2 - t_1} \int_{t_1}^{t_2} h(\tau) d\tau$.

Next suppose that $h(t) < h(t_0)$ for $t > t_0$ and $h(t) > h(t_0)$ for $t < t_0$, where $t \in N$. Since $G(t) = \int_{t_0}^{t} h(\tau) - h(t_0) d\tau$, it follows that there exist numbers c_1, d_1 in N such that $c_1 < t_0 < d_1$ and $G(c_1) < 0, G(d_1) < 0$. Note that $G(t_0) = 0$. Therefore, by the continuity of $G(t)$ and the Intermediate Value Theorem, it follows that there exist numbers t_1, t_2 in N such that $c_1 \le t_1 < t_0 < t_2 \le d_1$ and $G(t_1) = G(t_2)$. It immediately follows that
$$h(t_0) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} h(\tau) d\tau .$$
This completes the proof of the lemma.