Hausdorff Circles in Homogeneous Metric Continua

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Abstract

A brief history of the main theorem in this paper is presented. Due to some variation in the literature, basic relevant definitions and notations are also provided. Preliminary results are then developed for a particular class of homogeneous continua. It is shown that each point of an arbitrary open set in such a continuum is contained in a special type of subcontinuum, which is itself contained in the open set. Furthermore, each point of the space has a neighborhood base consisting of the same type of subcontinua. We then prove that any continuum in this class containing a subcontinuum with three point boundary must also contain a subcontinuum with two point boundary. It is then shown that all homogeneous metric continua which can be separated by either two or three points can be separated by each pair of points. Finally, we establish that all such continua are Hausdorff circles.

Introduction

In 1944 Gustave Choquet claimed that every homogeneous, compact plane continuum is necessarily a simple closed curve [3, pp. 542-544]. However, in 1948 Edwin E. Moise proved the existence of a compact plane continuum which is not an arc, but which is homeomorphic to each of its subcontinua [11, pp. 581-594]. Furthermore, R. H. Bing verified in 1948 that the continuum established by Moise is in fact homogeneous [1, pp. 729-742], thereby refuting the earlier claim by Choquet. The following year in 1949, F. Burton Jones provided two additional conditions [8, pp. 113-114], either of which added to the hypothesis of Choquet's paper, would validate the claim made by Choquet.

Pursuing these results, Forest W. Simmons showed in October 1980 that if a homogeneous continuum is separated by some pair of points, then it is separated by each pair of its points [12, pp. 62-73, Main Theorem]. Since it was shown by Winton that a homogeneous continuum cannot be separated by a single point [16, Theorem 2], then such continua are Hausdorff circles.

Several of Simmon's preliminary results were generalized in September 2007 by Richard A. Winton ([15, Lemma 2],[15, Corollary 5],[15, Theorem 6]). In February 2009 Winton established the precise conditions under which subcontinua with finite boundaries in a homogeneous continuum can be separated by a single point ([16, Theorem 5],[16, Theorem 6]). Winton then showed in September 2010 that the boundaries of a specific category of subcontinua in a certain class of homogeneous metric continua form a partition of the collection of all boundary points of the same class of subcontinua [17, Theorem 5]. Finally, in February 2017 it was verified by Winton that each point in a specific type of continuum is the intersection of a maximal chain from a particular class of subcontinua [18, Theorem 10] relative to a partial ordering defined by Winton which is dependent on the topology of the space [18, Definition 9].

We now proceed with definitions which are fundamental for the results that follow. Notations for these concepts are also provided since they are not uniform throughout the literature. For completeness, included in the following definitions is that of the above mentioned partial order on a topological space established by Winton in 2017 ([18, Lemma 8], [18, Definition 9]).

Basic Definitions

In the most general sense, a continuum is a compact, connected, Hausdorff topological space. In particular, a metric continuum is a compact, connected metrizable space. A Hausdorff circle is a Hausdorff topological space which cannot be separated by a single point, but which is separated by each pair of its points.

If H is a subset of a topological space X, then Int(H), Bd(H), and Cl(H) are the topological interior, boundary, and closure of H in X, respectively. A separation $A|B$ of H is a partition of H into nonempty relatively open sets A and B in H. Furthermore, H separates X if and only if X is connected but X−H is not connected. If n is a positive integer and X is connected, then n is the separation number of X, denoted by S(X), if and only if X contains a subset with n points which separates X, but X contains no subset with less than n points which separates X. In other words, S(X) is the minimal number of points required to separate X.

If n is an integer and $n > 1$, then H is an n-pod of X if and only if H is a subcontinuum of X whose boundary contains precisely n points. As special cases, 2-pods in X will be called bipods, while 3-pods in X will be referred to as tripods. An abipodic space is a topological space which contains no bipods. Furthermore, n is the pod number of X, denoted by P(X), if and only if X contains an n-pod but X contains no k-pod for each integer k with the property that $1 < k < n$.

Finally, suppose that X is a topological space and S is an arbitrary collection of subsets of X. The partial order ~ on S defined by A~B if and only if either
$A \subseteq \text{Int}(B)$ or $A = B$ for each $A,B \in S$ is called the ordering of S by interior inclusion ([18, Lemma 8],[18, Definition 9]).

Preliminary Results

In an abipodic homogeneous metric continuum X that contains a tripod, each point of an arbitrary open set is contained in a tripod which in turn is contained in that open set. In particular, every subset H of X has the property that each point in the interior of H is contained in a tripod which is contained in the interior of H.

Lemma 1: Suppose X is an abipodic homogeneous metric continuum which contains a tripod, V is an open set in X, and $p \in V$. Then there is a tripod A in X such that $p \in A \subseteq V$.

Proof: Since X is an abipodic homogeneous continuum which contains a tripod, then P(X) = 3 [18, Lemma 2(c)]. Consequently there is a maximal chain M of tripods in X relative to ordering by interior inclusion such that $\bigcap_{H \in M} H = \{p\}$ [18, Theorem 10]. Thus M is a nonempty nested collection of tripods in X for which $H \subseteq Int(K)$, $K \subseteq Int(H)$, or $H = K$ for each $H, K \in M$.

For each $H \in M$, H is a compact subset of the Hausdorff space X, so that H is closed in X ([2, p. 81, Corollary 5.13],[5, p. 165, Theorem 6.4]), and so X−H is open in X. Since $p \in V$ then $X - V \subseteq X - \{p\} = X - \bigcap_{H \in M} H = \bigcup_{H \in M} (X - H)$ by De Morgan's Laws, and so $S = \{X - H\}_{H \in M}$ is an open cover of X−V. Furthermore, since V is open in X then X−V is a closed subset of the compact space X, and so X−V is compact ([10, p. 162, Theorem 2.11],[13, p. 111, Theorem A]). Thus M contains a finite subset $\{H_i\}_{i=1}^{n}$ such that $X - V \subseteq \bigcup_{i=1}^{n} (X - H_i) = X - \bigcap_{i=1}^{n} H_i$ by De Morgan's Laws, and so $\bigcap_{i=1}^{n} H_i \subseteq V$.

However, since M is a nested collection, then there is some t, $1 \le t \le n$, such that $H_t \subseteq H_i$ for $1 \le i \le n$. Therefore $H_t = \bigcap_{i=1}^{n} H_i \subseteq V$. Finally, since $H_t \in M$ then $p \in \{p\} = \bigcap_{H \in M} H \subseteq H_t$. Consequently H_t is a tripod in X with the property that $p \in H_t \subseteq V$.

A special case of Lemma 1 will be useful. Specifically, when a subset of a space has nonempty interior, then each point in the interior of that subset is contained in a tripod which is, in turn, contained in that interior. Hence we have the following result.

Corollary 2: Suppose X is an abipodic homogeneous metric continuum which contains a tripod, $H \subseteq X$, and $p \in Int(H)$. Then there is a tripod A in X such that $p \in A \subseteq Int(H)$.

Proof: Int(H) is an open set containing p. Thus by Lemma 1 there is a tripod A in X such that $p \in A \subseteq Int(H)$.

Suppose that a compact Hausdorff topological space X contains a nested collection C of compact subsets. If the intersection of the sets in C is contained in an open set in X, then at least one of the subsets of X in C must also be contained in that open set.

Lemma 3: Suppose that C is a nested collection of compact sets in a compact Hausdorff topological space X, V is an open set in X, and $\bigcap_{H\in C} H \subseteq V$. Then $K \subseteq V$ for some $K\in C$.

Proof: For each $H\in C$, H is a compact subset of the Hausdorff space X. Therefore H is closed in X ([2, p. 81, Corollary 5.13],[5, p. 165, Theorem 6.4]), so that X−H is open in X for each $H\in C$. Since V is open in X as well, then $S = \{V\}\cup\{X-H\}_{H\in C}$ is a collection of open sets in X. Furthermore, since $\bigcap_{H\in C} H \subseteq V$ then $X-V \subseteq X-\bigcap_{H\in C} H = \bigcup_{H\in C}(X-H)$ by De Morgan's Laws, and so $X \subseteq \bigcup_{N\in S} N$. Therefore S is an open cover for X. Since X is compact, then S has a finite subset T for which $X \subseteq \bigcup_{N\in T} N$. Define $R = T-\{V\}$, so that R is a finite subset of $\{X-H\}_{H\in C}$. Consequently there is a finite subset $\{H_i\}_{i=1}^{n}$ of C such that $R = \{X-H_i\}_{i=1}^{n}$.

Since $X \subseteq \bigcup_{N\in T} N$ then $X-V \subseteq \bigcup_{N\in R} N = \bigcup_{i=1}^{n}(X-H_i) = X-\bigcap_{i=1}^{n} H_i$ by De Morgan's Laws, and so $\bigcap_{i=1}^{n} H_i \subseteq V$. However, since $\{H_i\}_{i=1}^{n} \subseteq C$ and C is a nested collection, then there is some t, $1 \le t \le n$, such that $H_t \subseteq H_i$ for $1 \le i \le n$. Hence $H_t = \bigcap_{i=1}^{n} H_i \subseteq V$.

We now begin the final approach to the main theorem. The next result establishes that in an abipodic homogeneous metric continuum containing a tripod, each point has a base for its neighborhood system consisting of a chain of tripods in X relative to ordering by interior inclusion. Furthermore, this chain is maximal in the collection of all tripods in X.

Lemma 4: Suppose X is an abipodic homogeneous metric continuum which contains a tripod. Then for each $p\in X$, there is a maximal chain in the collection of all tripods in X relative to ordering by interior inclusion which is a base for the neighborhood system of p.

Proof: Suppose $p\in X$. The set S of all tripods in X is partially ordered by interior inclusion ~ ([18, Lemma 8],[18, Definition 9]). Furthermore, $\{p\} = \bigcap_{H\in M} H$ for some maximal chain M in S relative to ~ [18, Theorem 10].

Assume that $p \in Bd(K)$ for some $K \in M$. Since $Int(K)$ is a nonempty open set, then by Lemma 1 or Corollary 2 there is a tripod B in X such that $B \subseteq Int(K)$, and so $B \sim K$. Suppose that $N \in M$. Since $K \in M$ as well and M is linearly ordered by $\sim$, then either $N \sim K$ or $K \sim N$. Thus either $N \subseteq Int(K)$, $K \subseteq Int(N)$, or $K = N$. However, if $N \subseteq Int(K) = K - Bd(K)$ ([9, p. 46, Theorem 10],[14, p. 28, Theorem 3.14(b)]), then $p \notin N$ since $p \in Bd(K)$. Since $N \in M$ then $p \notin \bigcap_{H \in M} H = \{p\}$, which is a contradiction. Therefore either $K \subseteq Int(N)$ or $K = N$. In either case we have $B \subseteq Int(K) \subseteq Int(N) = N - Bd(N)$ ([9, p. 46, Theorem 10],[14, p. 28, Theorem 3.14(b)]) $\subset N$ since N is a tripod. Consequently $B \sim N$ and $B \neq N$ for each $N \in M$. As a result, $M \cup \{B\}$ is a chain in S which properly contains M. However, this contradicts the maximality of M, and so $p \notin Bd(H)$ for each $H \in M$.

However, since $p \in \bigcap_{H \in M} H$, then $p \in H - Bd(H) = Int(H)$ ([9, p. 46, Theorem 10],[14, p. 28, Theorem 3.14(b)]) for each $H \in M$. Thus H is a neighborhood of p for each $H \in M$.

Now suppose U is an arbitrary neighborhood of p. Then there is an open set V in X such that $p \in V \subseteq U$. Since M is a nested collection of tripods and $\bigcap_{H \in M} H = \{p\} \subseteq V$, then $D \subseteq V \subseteq U$ for some $D \in M$ by Lemma 3. Hence M is a base for the neighborhood system of p.

Thus M is a maximal chain of tripods in S relative to ordering by interior inclusion which is a neighborhood base for p. Finally, since p was chosen arbitrarily in X (or since X is homogeneous), then each point of X has a neighborhood base consisting of a maximal chain of tripods in S relative to ordering by interior inclusion.

If a homogeneous metric continuum contains a tripod, then it must also contain a bipod. This result will now be confirmed in the contrapositive form. Thus we show that if a homogeneous metric continuum contains no bipods, then it cannot contain a tripod.

Theorem 5: An abipodic homogeneous metric continuum contains no tripods.

Proof: Suppose X is an abipodic homogeneous metric continuum. If X contains a tripod K then there exist $p,q,r \in X$ such that $Bd(K) = \{p,q,r\}$. Since X is Hausdorff then $\{q,r\}$ is closed in X ([7, p. 64, Corollary 3.12],[13, p. 130, Theorem A]), and so $X - \{q,r\}$ is an open set containing p. By Lemma 4 there is a tripod neighborhood H of p such that such that $H \subseteq X - \{q,r\}$, so that $\{q,r\} \cap H = \varnothing$. Furthermore, since H is a neighborhood of p then $p \in Int(H)$, and so $Bd(K) \cap Int(H) \supseteq \{p\} \neq \varnothing$.

Define $H' = Cl(X-H)$ and $K' = Cl(X-K)$. Then H' and K' are tripods in X with $Bd(H') = Bd(H)$ and $Bd(K') = Bd(K)$ [15, Theorem 6].

Assume that $Bd(H') \cap Int(K') = \varnothing$. Then $Bd(H) \cap (X-K) = \varnothing$ ([4, p. 142, Theorem 30.2],[6, p. 72, Theorem 4.11(4)]), so that $Bd(H) \subseteq K$. Since $p \in Bd(K)$

and $p \in Int(H) = H - Bd(H)$ ([9, p. 46, Theorem 10],[14, p. 28, Theorem 3.14(b)]) then $Bd(H) \neq Bd(K)$. Furthermore, since X is an abipodic homogeneous continuum and contains a tripod, then $P(X) = S(X) = 3$ [18, Lemma 2(c)].
Thus $Bd(H) \cap Bd(K) = \varnothing$ ([17, Lemma 4],[17, Theorem 5]). Since $Bd(H) \subseteq K$ and $Bd(H) \cap Bd(K) = \varnothing$ then $Bd(H) \subseteq K - Bd(K) = Int(K)$ ([9, p. 46, Theorem 10],[14, p. 28, Theorem 3.14(b)]), so that $K' = X - Int(K)$ ([4, p. 142, Theorem 30.2],[6, p. 72, Theorem 4.11(4)]) $\subseteq X - Bd(H) = Int(H) \cup Int(H')$ ([4, p. 142, Theorem 30.2],[6, p. 72, Theorem 4.11(4)]). Since $Int(H)$ and $Int(H')$ are mutually separated and K' is connected, then either $K' \subseteq Int(H)$ or $K' \subseteq Int(H')$ [14, p. 192, Corollary 26.6]. Since it was shown above that $p \in Bd(K) \cap Int(H)$, then $\varnothing \neq Bd(K) \cap Int(H) = Bd(K') \cap Int(H) \subseteq K' \cap Int(H)$. Therefore $K' \subseteq Int(H)$, and so $\{p,q,r\} = Bd(K) = Bd(K') \subseteq K' \subseteq Int(H) \subseteq H$. However, it was established above that $\{q,r\} \cap H = \varnothing$. This is a contradiction, and so $Bd(H') \cap Int(K') \neq \varnothing$.

Thus $\{H,H'\}$ and $\{K,K'\}$ are pairs of complementary tripods in X [15, Definition 7] such that $Bd(K) \cap Int(H) \neq \varnothing$ and $Bd(H') \cap Int(K') \neq \varnothing$. Since it was shown above that $P(X) = 3$, then $K \subseteq Int(H)$ [18, Lemma 6], so that $\{q,r\} \subseteq Bd(K) \subseteq K \subseteq Int(H) \subseteq H$. As above, however, this contradicts the previously established fact that $\{q,r\} \cap H = \varnothing$. Hence X contains no tripods.

We are now prepared to present the main result. A homogeneous metric continuum which can be separated by some subset of two or three points can, in fact, be separated by any two of its points. Consequently, the continuum is a Hausdorff circle.

Main Theorem

Theorem 6: If a homogeneous metric continuum X is separated by two or three of its points, then X is separated by each pair of its points. Consequently X is a Hausdorff circle.

Proof:
Case 1: If X is separated by two of its points, then X is separated by each pair of its points [12, p. 62, Main Theorem]. However, X is not separated by any single point [16, Theorem 2]. Consequently X is a Hausdorff circle.

Case 2: Now suppose that X is separated by three of its points, but X is not separated by any two of its points. Since X cannot be separated by a single point [16, Theorem 2], then $S(X) > 1$. Furthermore, since X is not separated by any two of its points, then $S(X) > 2$. Therefore X contains no bipods by the contrapositive of [15, Lemma 3], and so X is abipodic. Since $S(X) > 2$ and X is separated by three of its points, then $S(X) = 3$ and there exist $p,q,r \in X$ such that $\{p,q,r\}$ separates X. As a result there is a separation $A|B$ of $X - \{p,q,r\}$. Thus $A \cup \{p,q,r\}$ and $B \cup \{p,q,r\}$ are tripods in X [15, Lemma 2]. Consequently, X is an abipodic homogeneous metric continuum which contains a tripod, which contradicts Theorem 5.

Thus if X is separated by three of its points, then X is separated by two of its points. Hence X is separated by each pair of its points [12, p. 62, Main Theorem]. Since X is not separated by any single point [16, Theorem 2], then X is a Hausdorff circle.

Alternate proof of Case 2:

Suppose that X is separated by three of its points. Then $S(X) \leq 3$ and there exist $p,q,r \in X$ such that $\{p,q,r\}$ separates X. Thus there exists a separation $A|B$ of $X - \{p,q,r\}$.

Assume that $S(X) = 3$. Therefore $A \cup \{p,q,r\}$ and $B \cup \{p,q,r\}$ are tripods in X [15, Lemma 2], so that $P(X) = S(X) = 3$ [15, Corollary 4]. Since X contains tripods, then X contains a bipod by the contrapositive of Theorem 5, so that $P(X) \leq 2$. However, this contradicts the above conclusion that $P(X) = 3$, and so $S(X) \neq 3$.

Since it was established above that $S(X) \leq 3$, then $S(X) \leq 2$. However, since X cannot be separated by a single point [16, Theorem 2], then $S(X) \geq 2$. Therefore $S(X) = 2$, so that X is separated by some pair of its points. Consequently X is separated by each pair of its points [12, p. 62, Main Theorem]. Furthermore, since no single point separates X [16, Theorem 2], then X is a Hausdorff circle.

As a result of Theorem 6 (or Theorem 5), we have a somewhat surprising consequence. The result is the final Corollary.

Corollary 7: There exist no homogeneous metric continua with pod number 3.

Proof: Suppose that X is a homogeneous metric continuum with $P(X) = 3$. Then X contains a tripod H with $Bd(H) = \{p,q,r\}$ for some $p,q,r \in X$. Since X contains the tripod H, then $S(X) = P(X) = 3$ [15, Corollary 4]. Furthermore, Bd(H) separates X [15, Lemma 3]. Since X is separated by $\{p,q,r\}$, then X is a Hausdorff circle by Theorem 6. Therefore X is separated by $\{p,q\}$, so that $S(X) \leq 2$. (In fact, since is has been established that X cannot be separated by a single point [16, Theorem 2], then $S(X) = 2$.)

This contradicts the fact established above that $S(X) = 3$. Hence there exist no homogeneous metric continua with $P(X) = 3$.

Alternate proof: If X is a homogeneous metric continuum with $P(X) = 3$, then X contains a tripod. Therefore X contains a bipod by (the contrapositive of) Theorem 5, and so $P(X) \leq 2$ by the definition of pod number.

However, this contradicts the hypothesis that $P(X) = 3$. Hence there exist no homogeneous metric continua with $P(X) = 3$.

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