# The problem of the most visited point and novel properties of Pascal's Triangle

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**Abstract**

In this article, we will find the point of the Cartesian plane through which more paths pass in a quadric mesh of paths starting from point A = (0,0) and arriving at B = (*N*,*N*). It is only allowed to go right or up. In our journey, we will be rewarded for the visualization of two properties of the Pascal's Triangle recently discovered. One of them says that the product of center elements of two lines that have an odd number of elements is larger than the product of central elements of lines that also have an odd number of elements that are symmetrically located between the two. We based on the Melo, Santos (2014) and Santos, Castilho (2013) works. Only finite induction was used for the proves.

**Keywords:** Pascal's Triangle. Combinatorial Analysis. Most visited point. Paths in the plane.



**Stating the problem**

Consider the grid system of paths in the figure below starting from point A = (0,0) and arriving at B = (3,3). It is only allowed to go right or up.

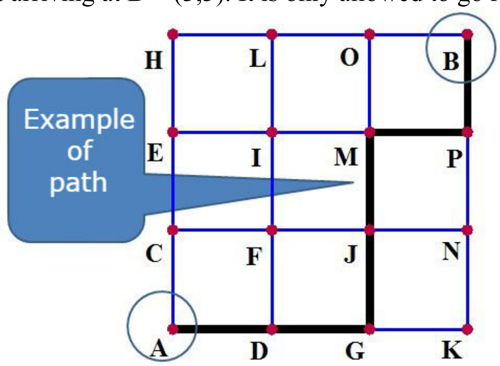



Figure 1 – 3 by 3 Grid of paths

Because the arrival point is (3,3), we will call this lattice 3 by 3.

In the above case there is a total of $\frac{(3+3)!}{3!3!} = 20$ paths starting from A = (0,0) and arriving at B = (3,3), because, of the 6 steps that must be taken, 3 must be taken up and 3 to the right, in any order. Therefore, the total number of paths is the combination of 6 steps taken 3 at a time.

Note that by the rules that the path must follow, only one path passes through the coordinate point (3,0). Also, a single path passes through the point (0,3). For the other points, there are more paths passing through each of them. For example, the following number of paths passes through the coordinates point (1,2):

$$\underbrace{\frac{(1+2)!}{1!\,2!}}_{\substack{\text{paths that}\\ \text{arrive}\\ \text{at pint}\\ (1,2)}} \times \underbrace{\frac{(2+1)!}{2!\,1!}}_{\substack{\text{paths that leave}\\ \text{point } (1,2) \text{ and}\\ \text{arrive at point}\\ (3,3)}} = 3 \times 3 = 9.$$

The question we will raise in our study is: without taking the points (0,0) and (3,3) into account, through what point does the greatest number of paths starting from (0,0) and arriving at (3,3) pass? And in an *N* by *N* square of any order? To answer this question, we need the following considerations:

**Number of paths arriving at a given point**

From the origin (0,0) to some point of type (0, i) or (i, 0), with i = 0, 1, 2, 3, only one path *arrives*, where, for convenience, we will consider that also *only one path arrives* at point (0,0).

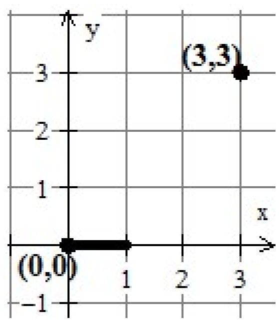



Figure 2 – Only one path *arrives* at point (1,0)

Thus, two paths come from the origin to point (1,1): the one that comes from (1,0) and the one that comes from (0,1). Also, from the origin to point (1,2) there are 3 paths: two coming from (1,1) and one coming from (0,2):

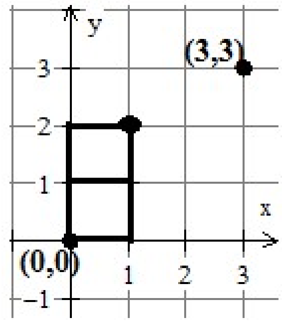





Figure 3 – Three paths arrive at point (1,2): one coming from the left and two coming from below.

Generalizing, the paths coming from ($m$ - 1, $n$) and ($m$, $n$ - 1) arrive at point ($m$, $n$), with $m$> 0 and $n$> 0, as shown in matrix A below, where each element of position ($m$, $n$) corresponds to the number of paths coming from the origin to ($m$, $n$): A =

$$\begin{pmatrix} 1 & 1+3=4 & 6+4=10 & 10+10=20 \\ 1 & 1+2=3 & 3+3=6 & 4+6=10 \\ 1 & 1+1=2 & 1+2=3 & 1+3=4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 10 & 20 \\ 1 & 3 & 6 & 10 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

,

Note that, therefore, matrix A consists of elements of the Pascal's Triangle, in ***bold*** below.

**1**
**1 1**
**1 2 1**
**1 3 3 1**
1 **4 6 4** 1
1 5 **10 10** 5 1
1 6 15 **20** 15 6 1

We can do the same analysis for the number of paths that leave each point ($m$, $n$) and arrive at (3,3), which is the same thing as leaving (3,3) and arriving at ($m$, $n$). In this case, the matrix will be the transposed matrix of the previous one, also symmetrical in relation to the secondary diagonal: $A^t =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 \\ 10 & 6 & 3 & 1 \\ 20 & 10 & 4 & 1 \end{pmatrix}$$

Finally, to obtain the number of paths that pass through the point ($m$, $n$) with $m$ = 0, 1, 2, 3 and $n$ = 0, 1, 2, 3, we simply multiply the number of paths arriving at ($m$, $n$) by the number of paths that leave ($m$, $n$), that is, we simply multiply the corresponding elements of the two matrices A and $A^t$ above. The result will also be a symmetric matrix, which we call B:

$$\begin{pmatrix} 1 & 4 & 10 & 20 \\ 4 & 9 & 12 & 10 \\ 10 & 12 & 9 & 4 \\ 20 & 10 & 4 & 1 \end{pmatrix}$$



Finally, looking at matrix B, we conclude that the points (1,1) and (2,2) are the points of the mesh through which more paths pass, from the origin to (3,3): 12 paths in total, through each one.

Note that because we are multiplying here element by element of two transposed and symmetric matrices, we have that the element $b_{m,n}$ of matrix B equals the following products formed by elements of matrix A: $a_{m,n} \times a_{3-m,3-n}$, or $a_{m,n} \times a_{3-n,3-m}$, or, alternatively, $a_{n,m} \times a_{3-m,3-n}$, or, even, $a_{n,m} \times a_{3-n,3-m}$.

**First proposition: of the symmetry of matrix B**

In an $N$ by $N$ grid system, let us call c ($n$, $m$) the number of paths that leave (0,0) and arrive at point ($N$, $N$), passing through ($n$, $m$). The first proposition that we are going to demonstrate, which can be inferred from the previous example, is: c ($n$, $m$) = c ($m$, $n$) = c ($N - n$, $N$-$m$).

*Demonstration: The number of paths from (0,0) to (n, m) is given by the formula for permutation with repetition,*$P^{n+m}_{n,m}$.The number of paths from ($n$, $m$) to ($N$,$N$) is, in turn, $P^{N-n+N-m}_{N-n,N-m}$. Now, let us observe that $c(n,m)$ *will be the product of the number of paths arriving in (n,m) by the number of paths starting from (n,m).* Thus,

$$c(n,m) = P^{n+m}_{n,m} \times P^{2N-n-m}_{N-n,N-m}$$

paths. Note that this formula is symmetric in relation to $n$ and $m$, so that it is clear that c ($m$, $n$) = c ($n$, $m$).

Now, from (0,0) to ($N - n$, $N - m$) there are $P^{N-n+N-m}_{N-n,N-m}$ paths. And from ($N - n$, $N - m$) to ($N$, $N$) there are $P^{n+m}_{n,m}$ paths. Thus, $c(N - n, N - m) = c(n,m)$ paths, and, according to the first step of this demonstration, it also equals $c(N - m, N - n)$, which concludes the demonstration.

**Second proposition: of the diagonal of the matrix**

The second proposition, which can be inferred from the results of the secondary diagonal of the matrix of our example, namely 20, 12, 12, 20, is: for $0 \leq x < N/2$,

$$c(x+1, x+1) \leq c(x, x.)$$

That is, the number of paths decreases when we take points farther away from the origin, at least up to half the diagonal. In our case, from 20 to 12. Moreover, for $N/2 < x \leq N$, let $c(x - 1, x - 1) \leq c(x, x)$. That is, the number of paths increases when we get points closer and closer to $(N, N)$. In our case, from 12 to 20. This is true for any even or odd positive $N$.

*Demonstration*: $\underbrace{P^{2x}_{x,x}}_{arrive\ at\ P} \times \underbrace{P^{2N-2x}_{N-x,N-x}}_{leave\ P}$ paths pass through P = $(x, x)$, where $0 \leq x < N/2$, going from (0,0) to (N, N).

$\underbrace{P^{2x+2}_{x+1,x+1}}_{arrive\ at\ Q} \times \underbrace{P^{2N-2x-2}_{N-x-1,N-x-1}}_{leave\ Q}$ paths, in turn, pass through Q = $(x + 1, x + 1)$

Let us suppose by contradiction that:

$$P^{2x+2}_{x+1,x+1} \times P^{2N-2x-2}_{N-x-1,N-x-1} > P^{2x}_{x,x} \times P^{2N-2x}_{N-x,N-x}.$$

Then:

$$\frac{(2x+2)!}{(x+1)!^2} \cdot \frac{(2N-2x-2)!}{(N-x-1)!^2} > \frac{(2x)!}{x!^2} \cdot \frac{(2N-2x)!}{(N-x)!^2} \xRightarrow[developing\ and\ simplifying\ the\ factorials]{}$$

$$\frac{(2x+2)(2x+1)}{(x+1)^2} > \frac{(2N-2x)(2N-2x-1)}{(N-x)^2} \Rightarrow$$

$$\frac{2(2x+1)}{x+1} > \frac{2(2N-2x-1)}{N-x} \Rightarrow$$

$$\frac{2x+1}{x+1} > \frac{2N-2x-1}{N-x} \xRightarrow[N>x]{} (2x+1)(N-x) > (2N-2x-1)(x+1) \Rightarrow$$

$$2xN - 2x^2 + N - x > 2Nx + 2N - 2x^2 - 2x - x - 1 \Rightarrow$$

$$-N > -2x - 1 \Rightarrow$$

$$x > \frac{N-1}{2} = \frac{N}{2} - 0{,}5 \Rightarrow x \geq \frac{N}{2},$$

where the latter implication would be valid for even or odd $N$ because x is integer. So, a contradiction.

Now, for $N/2 < x \leq N$, we expect, the reader can observe that $c(x - 1, x - 1) \leq c(x, x)$ only observing the symmetry between the points of the matrix, which we have already proved in the first proposition. But for those who wish to rigorously demonstrate this second part, it is enough to follow steps similar to

those made in the first part of this proposition, or else define y = *N* - x, and hence work with the fact that *c*(*x*, *x*) = *c*(*y*, *y*), which was already proved in the first proposition, and apply in y what we have just proved in the first part of this second proposition.
Note also that, according to the above calculations, c(x + 1, x + 1) = c(x, x) if and only if $x = \frac{N-1}{2}$, that is, *N* must be odd. In this case,

$c\left(\frac{N+1}{2}, \frac{N+1}{2}\right) = c\left(\frac{N-1}{2}, \frac{N-1}{2}\right)$, the two central points of the secondary diagonal, as was the case in our previous example, where c(1,1) = c(2,2) = 12.
Note, therefore, that this second proposition is saying that, on the diagonal, the points that have more paths passing through them are the points (1,1) and, by symmetry, $(N-1, N-1)$, where we exclude, of course, the points (0,0) and (*N*, *N*), which are of mandatory passage.

**Third proposition: of the points below and above the diagonal**

Let us finally prove our third and final proposition. It will tell us that
$c(x,x) \geq c(x, x-i)$
for every x = 1, ..., N, with *i* = 0, ..., x. By symmetry, again, the reader can easily be convinced that, consequently, $c(x,x) \geq c(x, x+i)$ for all x = 1, ..., *N*, with *i* = 0, ..., *N* - x. That is, in each column of the path mesh, the point though which most paths pass is the point belonging to the secondary diagonal, as can be seen in matrix B of our example.
*Demonstration*: We will do it by induction on *i*. For *i* = 0, it is easy to see that equality is verified. For *N* = 1, it is also easy to see that c(0,0) = c(1,1) = 2 and c(1,0) = c(0,1) = 1. Let us suppose then, by induction hypothesis, that, for *i* > 0, we have *c*(*x*,*x*) > *c*(*x*, *x* – *i)* (by proving the restricted inequality we will have proved the unrestricted inequality as an immediate consequence).

Thus,

$$c(x,x) > c(x,x-i) \Leftrightarrow P^{2x}_{x,x} \cdot P^{2N-2x}_{N-x,N-x} > P^{2x-i}_{x,x-i} \cdot P^{2N-2x+i}_{N-x,N-x+i} \Rightarrow$$
$$\frac{(2x)!}{(x!)^2} \cdot \frac{(2N-2x)!}{(N-x)!^2} > \frac{(2x-i)!}{x!\,(x-i)!} \cdot \frac{(2N-2x+i)!}{(N-x)!\,(N-x+i)!}$$

Completing the induction process, let us prove $c(x,x) > c(x, x-(i+1))$:

$$c(x, x-(i+1)) = \frac{(2x-i-1)!}{x!\,(x-i-1)!} \cdot \frac{(2N-2x+i+1)!}{(N-x)!\,(N-x+i+1)!} =$$
$$\frac{1}{x!} \cdot \frac{(2x-i)!}{(2x-i)} \cdot \frac{(x-i)}{(x-i)!} \cdot \frac{(2N-2x+i+1)(2N-2x+i)!}{(N-x)!\,(N-x+i+1)(N-x+i)!} =$$
$$\left[\frac{(2x-i)!}{x!\,(x-i)!} \cdot \frac{(2N-2x+i)!}{(N-x)!\,(N-x+i)!}\right]$$
$$\cdot \frac{(2N-2x+i+1)(x-i)}{(N-x+i+1)(2x-i)} \underset{induction\ hyphotesis}{<}$$

$$\left[\frac{(2x)!}{(x!)^2}\cdot\frac{(2N-2x)!}{(N-x)!^2}\right]\cdot\left[\frac{(2N-2x+i+1)(x-i)}{(N-x+i+1)(2x-i)}\right]=$$
$$c(x,x)\cdot\left[\frac{(2N-2x+i+1)(x-i)}{(N-x+i+1)(2x-i)}\right],$$

which is smaller than $c(x,x)$, as we will now prove:

$$\frac{(2N-2x+i+1)(x-i)}{(N-x+i+1)(2x-i)}<1 \xLeftrightarrow[N\geq x \text{ and } i\leq x<2x]{}$$
$$2Nx-2Ni-2x^2+2xi+ix-i^2+x-i$$
$$<2Nx-Ni-2x^2+ix+2ix-i^2+2x-i \Leftrightarrow$$
$$-Ni+x<2x \Leftrightarrow$$
$$i>-\frac{x}{N}\geq -1,$$

because $x \leq N$, and this latter inequality is more than we needed, since, in our case, $i = 1, ..., x$. So, our proposition is proved.

**Finally, the point sought**

Now, by bringing together the conclusions of the last two propositions, we conclude that in any $N$ by $N$ grid of paths, the points through which most paths pass are the points (1,1) and ($N$ - 1, $N$ - 1), without taking into account, of course, the points (0,0) and ($N$, $N$):

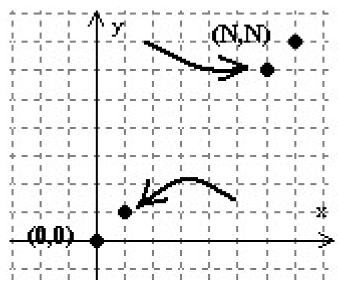
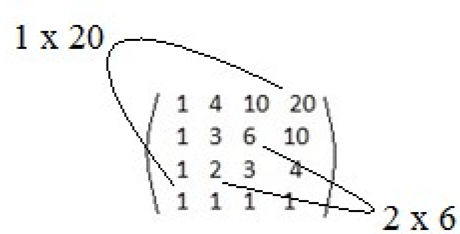



Figure 4 – The most visited points

**Novel properties of Pascal's Triangle**

According with the second proposition, at the matrix below, which shows a particular case, we have: $1 \times 20 > 2 \times 6$:

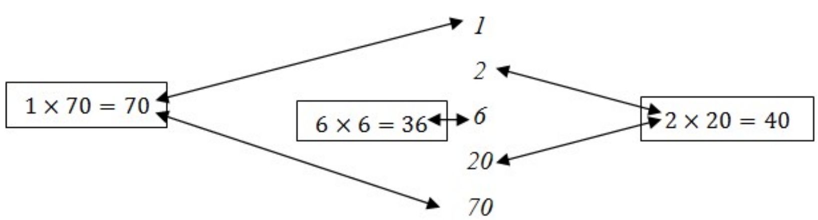


What the second proposition of this article tells us is that by looking at the Pascal's Triangle from the perspective below, the product of center elements of two lines that have an odd number of elements is larger than the product of central elements of lines that also have an odd number of elements that are symmetrically located between the two. In our example, we have:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | |
| 1 | 1 | | | | | | | | | | |
| 1 | 2 | 1 | | | | | | | | | |
| 1 | 3 | 3 | 1 | | | | | | | | |
| 1 | 4 | 6 | 4 | 1 | | | | | | | |
| 1 | 5 | 10 | 10 | 5 | 1 | | | | | | |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | | | |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | | |
| 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | | |
| 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | | |
| 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 | |
| 1 | 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 | 11 | 1 |

Figure 5 – Novel property of Pascal's Triangle: 1 x 20 = 20 > 2 x 6 = 12.

Looking only at the central column of the Triangle, we also have:

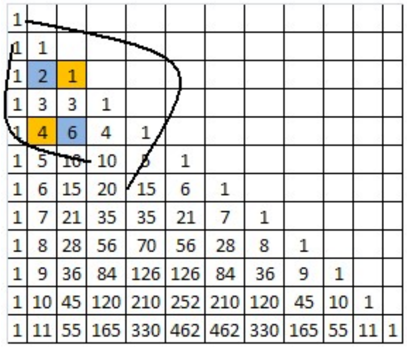



*70 > 40 > 36*

Figure 6 – In the central column, 70 > 40 > 36

The third proposition, in turn, tells us that, looking at the Pascal's Triangle from the same perspective below, and given the integer k, the product of two central elements of two lines *i* and *j* is greater than the product of the elements that are in the same columns and in lines *i* + *k*, and *j* - *k*, as in our example, where we have:

2 x 6 = 12 > 4 x 1 = 4. Or also 1 x 20 = 20 > 1 x 10 = 10:

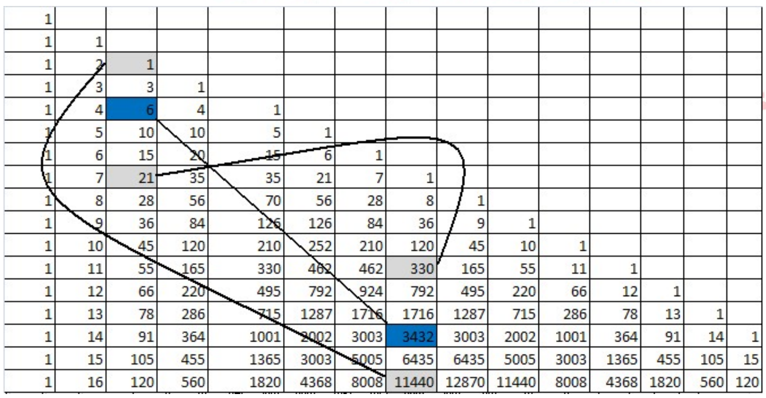


| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | |
| 1 | 1 | | | | | | | | | | |
| 1 | 2 | 1 | | | | | | | | | |
| 1 | 3 | 3 | 1 | | | | | | | | |
| 1 | 4 | 6 | 4 | 1 | | | | | | | |
| 1 | 5 | 10 | 10 | 5 | 1 | | | | | | |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | | | |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | | |
| 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | | |
| 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | | |
| 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 | |
| 1 | 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 | 11 | 1 |

Figure 7 – Another novel property of Pascal's Triangle



Or also, in the matrix below: 6 x 3.432 = 20.592 > 21 x 330 = 6.930, and also greater than 1 x 11.440:



| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | | | |
| 1 | 1 | | | | | | | | | | | | | |
| 1 | 2 | 1 | | | | | | | | | | | | |
| 1 | 3 | 3 | 1 | | | | | | | | | | | |
| 1 | 4 | 6 | 4 | 1 | | | | | | | | | | |
| 1 | 5 | 10 | 10 | 5 | 1 | | | | | | | | | |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | | | | | | |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | | | | | |
| 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | | | | | |
| 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | | | | | |
| 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 | | | | |
| 1 | 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 | 11 | 1 | | | |
| 1 | 12 | 66 | 220 | 495 | 792 | 924 | 792 | 495 | 220 | 66 | 12 | 1 | | |
| 1 | 13 | 78 | 286 | 715 | 1287 | 1716 | 1716 | 1287 | 715 | 286 | 78 | 13 | 1 | |
| 1 | 14 | 91 | 364 | 1001 | 2002 | 3003 | 3432 | 3003 | 2002 | 1001 | 364 | 91 | 14 | 1 |
| 1 | 15 | 105 | 455 | 1365 | 3003 | 5005 | 6435 | 6435 | 5005 | 3003 | 1365 | 455 | 105 | 15 |
| 1 | 16 | 120 | 560 | 1820 | 4368 | 8008 | 11440 | 12870 | 11440 | 8008 | 4368 | 1820 | 560 | 120 |

Figure 8 – The novel property of Pascal's Triangle for other elements

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