Some Remarks on Generating Pythagorean Triples

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Abstract

In this paper, we propose a single-parameter based formulas to generate Pythagorean Triples and have used mathematical induction to prove them. We have also investigated triangles with integers sides whose area is an integer multiple of its perimeter.

Introduction

A Pythagorean Triple (a, b, c) consists of three positive integers a, b, and c, such that $a^2 + b^2 = c^2$ [1]. It is easy to see that for any positive integer $\lambda \in N$ a triple (a, b, c) is a Pythagorean triple (PT) if and only if (λ a, λ b, λ c) is also a Pythagorean triple. If all the three sides of a triangle form a Pythagorean triple then it is called a Pythagorean triangle and essentially a right triangle.
A primitive Pythagorean Triple (PPT) is a Pythagorean triple (a, b, c) in which the three integers a, b, and c are coprime i. e. they have no common divisor other than 1. In this paper, we have developed a method to generate PT based on a simple parameter $k \in N$.

Let us consider a general view of a Pythagorean triple (a, b, c) with $a < b < c$, and $c = b + s$, being another positive integer.

a) For $s = 1$, an odd integer, we can start probing since the smallest values for a and b, analyzing $a^2 + b^2 >\!<\!= c^2$ until we find those values that satisfy the Pythagorean Theorem, among others these:

a	b	s	c = b + s	a^2	b^2	C^2
3	4	1	5	9	16	25
5	12	1	13	25	144	169
7	24	1	25	49	576	625
9	40	1	41	81	1600	1681
11	60	1	61	121	3600	3721
13	84	1	85	169	7056	7225
15	112	1	113	225	12544	12769
17	144	1	145	289	20736	21025
…	…	…	…	…	…	…

Table (1)

These a, b, c represent Primitive Pythagorean Triples. From these results we get the conclusion that they can be well described in the formula:

$$(2k+1)^2 + [2k(k+1)]^2 = [2k(k+1)+1]^2, \text{ where } k \in \mathrm{N} \qquad (1)$$

Applying the Principle of Mathematical Induction: to prove equation (1)

For k=1 results $3^2 + 4^2 = 5^2$

Hypothesis: $(2k+1)^2 + [2k(k+1)]^2 = [2k(k+1)+1]^2$

Thesis: $[2(k+1)+1]^2 + [2(k+1)(k+2)]^2 = [2(k+1)(k+2)+1]^2$

$$[(2k+1)+2]^2 + [2k(k+1)+4(k+1)]^2$$
$$= [(2k(k+1)+1)+4(k+1)]^2$$
$$(2k+1)^2 + 4(2k+1) + 4 + [2k(k+1)]^2 + 8(k+1)(2k(k+1)) + 16(k+1)^2$$
$$= (2k(k+1)+1)^2 + 8(k+1)[2k(k+1)+1] + 16(k+1)^2$$
$$4(2k+1) + 4 + 8(k+1)(2k(k+1)) = 8(k+1)[2k(k+1)+1]$$
$$4(2k+1) + 4 = 8(k+1) \qquad \blacksquare$$

a) For $s = 2$, an even integer, we can start probing since the smallest values for a and b, analyzing $a^2 + b^2 >\!<\!= c^2$ until we find those values that satisfy the Pythagorean Theorem, among others these:

a	b	s	c = b + s	a^2	b^2	C^2
6	8	2	10	36	64	100
8	15	2	17	64	225	289
10	24	2	26	100	576	676
12	35	2	37	144	1225	1369
14	48	2	50	196	2304	2500
16	63	2	65	256	3969	4225
18	80	2	82	324	6400	6724
20	99	2	101	400	9801	10201
…	…	…	…	…	…	…

Table 2

From table 2, we get the conclusion that they can be well described in the formula: $[2(k+2)]^2 + [(k+1)(k+3)]^2 = [(k+1)(k+3)+2]^2$,

where $k \in \mathrm{N}$ (2)

Applying the Principle of Mathematical Induction: to prove equation (2)

For k=1 results $6^2 + 8^2 = 10^2$

Hypothesis:

$$\left[2\left(k+2\right)\right]^2+\left[\left(k+1\right)\left(k+3\right)\right]^2=\left[\left(k+1\right)\left(k+3\right)+2\right]^2$$

Thesis: $[2(k+3)]^2+[(k+2)(k+4)]^2=[(k+2)(k+4)+2]^2$

$$[2(k+2)+2]^2+[(k+1)(k+3)+(2k+5)]^2$$
$$=[((k+1)(k+3)+2)+(2k+5)]^2$$
$$(2(k+2))^2+8(k+2)+4+[(k+1)(k+3)]^2+2(2k+5)(k+1)(k+3)$$
$$+(2k+5)^2$$
$$=((k+1)(k+3)+2)^2+2(2k+5)(k+1)(k+3)$$
$$+4(2k+5)+(2k+5)^2$$
$$8(k+2)+4=4(2k+5) \qquad \blacksquare$$

b) For $s=3$, an odd integer, we can start probing since the smallest values for a and b, analyzing $a^2+b^2><=c^2$ until we find those values that satisfy the Pythagorean Theorem, among others these:

a	b	s	c = b + s	a^2	b^2	C^2
9	12	3	15	81	144	225
15	36	3	39	225	1296	1521
21	72	3	75	441	5184	5625
27	120	3	123	729	14400	15129
33	180	3	183	1089	32400	33489
39	252	3	255	1521	63504	65025
45	336	3	339	2025	112896	114921
51	432	3	435	2601	186624	189225
…	…	…	…	…	…	…

Table 3

From table 3, we get the conclusion that they can be well described in the formula:

$$\left[3\left(2k+1\right)\right]^2+\left[3\left[2k\left(k+1\right)\right]\right]^2=\left[3\left[2k\left(k+1\right)+1\right]\right]^2, \text{ where } k\in \mathrm{N} \qquad (3)$$

Applying the Principle of Mathematical Induction: to prove equation (3)

For k=1 results $9^2+12^2=15^2$

Hypothesis: $\left[3\left(2k+1\right)\right]^2+\left[3\left(2k\left(k+1\right)\right)\right]^2=\left[3\left(2k\left(k+1\right)+1\right)\right]^2$

Thesis:

$$[3(2k+3)]^2+\left[3\left(2(k+1)\right)(k+2)\right]^2=[3(2(k+1)(k+2)+1)]^2$$
$$[3(2k+1)+6]^2+[3(2k)(k+1)+12(k+1)]^2$$
$$=[3(2k(k+1)+1)+12(k+1)]^2$$

$$[3(2k+1)]^2 + 36(2k+1) + 36 + [3(2k(k+1))]^2 + 144\,k\,(k+1)^2 + 144(k+1)^2$$
$$= [3(2k(k+1)+1)]^2 + 144\,k\,(k+1)^2 + 72(k+1) + 144(k+1)^2$$
$$36(2k+1) + 36 = 72(k+1) \quad \blacksquare$$

c) For $s = 4$, an even integer, we can start probing since the smallest values for a and b, analyzing $a^2 + b^2 >\!<\!= c^2$ until we find those values that satisfy the Pythagorean Theorem, among others these:

a	b	s	c = b + s	a2	b2	C2
12	16	4	20	144	256	400
16	30	4	34	256	900	1156
20	48	4	52	400	2304	2704
24	70	4	74	576	4900	5476
28	96	4	100	784	9216	10000
32	126	4	130	1024	15876	16900
36	160	4	164	1296	25600	26896
40	198	4	202	1600	39204	40804
…	…	…	…	…	…	…

Table 4

From these results we get the conclusion that they can be well described in the formula:

$$[2\,[2\,(k+2)]]^2 + [2\,(k+1)\,(k+3)]^2 = [2\,[(k+1)\,(k+3)+2]]^2,$$

where $k \in \mathrm{N}$ (4)

Applying the Principle of Mathematical Induction: to prove equation (4)

For k=1 results $12^2 + 16^2 = 20^2$

Hypothesis:

$$[4\,(k+2)]^2 + [2\,(k+1)\,(k+3)]^2 = [2\,((k+1)\,(k+3)+2)]^2$$

Thesis: $[4(k+3)]^2 + [2\,(k+2)(k+4)]^2 = [2((k+2)(k+4)+2)]^2$

$$[4(k+2)+4]^2 + [2(k+1)(k+3) + 2(2k+5)]^2 = [2((k+1)(k+3)+2) + 2(2k+5)]^2$$

$$(4(k+3))^2 + 32(k+2) + 16 + [2(k+1)(k+3)]^2 + 8(k+1)(k+3)(2k+5) + 4(2k+5)^2$$
$$= [2((k+1)(k+3)+2)]^2 + 8((k+1)(k+3)+2)\,(2k+5)$$

$$32(k+2)+16 = 16(2k+5)$$

d) For $s = 5$, an odd integer, we can start probing since the smallest values for a and b, analyzing $a^2 + b^2 >\!<\!= c^2$ until we find those values that satisfy the Pythagorean Theorem, among others these:

a	b	s	c = b + s	a^2	b^2	C^2
15	20	5	25	225	400	625
25	60	5	65	625	3600	4225
35	120	5	125	1225	14400	15625
45	200	5	205	2025	40000	42025
55	300	5	305	3025	90000	93025
65	420	5	425	4225	176400	180625
75	560	5	565	5625	313600	319225
85	720	5	725	7225	518400	525625
…	…	…	…	…	…	…

Table 5

From table 5, we get the conclusion that they can be well described in the formula:

$$[5(2k+1)]^2 + [5[2k(k+1)]]^2 = [5[2k(k+1)+1]]^2,$$ where $k \in \mathrm{N}$ (5)

Applying the Principle of Mathematical Induction: to prove equation (5)

For k=1 results $15^2 + 20^2 = 25^2$

Hypothesis: $[5(2k+1)]^2 + [5(2k(k+1))]^2 = [5(2k(k+1)+1)]^2$

Thesis:

$$[5(2k+3)]^2 + [5(2(k+1))(k+2)]^2 = [5(2(k+1)(k+2)+1)]^2$$

$$[5(2k+1)+10]^2 + [5(2k)(k+1) + 20(k+1)]^2 = [5(2k(k+1)+1) + 20(k+1)]^2$$

$$[5(2k+1)]^2 + 100(2k+1) + 100 + [5(2k(k+1))]^2 + 400\,k(k+1)^2 + 400(k+1)^2 = [5(2k(k+1)+1)]^2 + 200\,[2k(k+1)+1](k+1) + 400(k+1)^2$$

$$100(2k+1) + 100 = 200(k+1)$$

a) For $s = 6$, an even integer, we can start probing since the smallest values for a and b, analyzing $a^2 + b^2 >< = c^2$ until we find those values that satisfy the Pythagorean Theorem, among others these:

a	b	s	c = b + s	a^2	b^2	C^2
18	24	6	30	324	576	900
24	45	6	51	576	2025	2601
30	72	6	78	900	5184	6084
36	105	6	111	1296	11025	12321
42	144	6	150	1764	20736	22500
48	189	6	195	2304	35721	38025
54	240	6	246	2916	57600	60516
60	297	6	303	3600	88209	91809
…	…	…	…	…	…	…

Table 6

From table 6, we get the conclusion that they can be well described in the formula:

$$[3[2(k+2)]]^2 + [3(k+1)(k+3)]^2 = [3[(k+1)(k+3)+2]]^2,$$

where $k \in \mathbb{N}$ (6)

Applying the Principle of Mathematical Induction: to prove equation (6)

For k=1 results $18^2 + 24^2 = 30^2$

Hypothesis:

$$[3(2(k+2))]^2 + [3(k+1)(k+3)]^2 = [3((k+1)(k+3)+2)]^2$$

Thesis: $[3(2(k+3))]^2 + [3(k+2)(k+4)]^2 = [3((k+2)(k+4)+2)]^2$

$$[3(2(k+2))+6]^2 + [3(k+1)(k+3)+3(2k+5)]^2$$
$$= [3((k+1)(k+3)+2)+3(2k+5)]^2$$

$$(3(2(k+2)))^2 + 72(k+2) + 36 + [3(k+1)(k+3)]^2$$
$$+ 18(k+1)(k+3)(2k+5) + 9(2k+5)^2$$
$$= [3((k+1)(k+3)+2)]^2$$
$$+ 18((k+1)(k+3)+2)(2k+5) + 9(2k+5)^2$$
$$72(k+2) + 36 = 36(2k+5) \quad \blacksquare$$

b) For $s = 7$, an odd integer, we can start probing since the smallest values for a and b, analyzing $a^2 + b^2 >\!<\!= c^2$ until we find those values that satisfy the Pythagorean Theorem, among others these:

a	b	s	c = b + s	a^2	b^2	C^2
21	28	7	35	441	784	1225
35	84	7	91	1225	7056	8281
49	168	7	175	2401	28224	30625
63	280	7	287	3969	78400	82369
77	420	7	427	5929	176400	182329
91	588	7	595	8281	345744	354025
105	784	7	791	11025	614656	625681
119	1008	7	1015	14161	1016064	1030225
…	…	…	…	…	…	…

Table 7

From table 7, we get the conclusion that they can be well described in the formula:

$$[7\ (2k+1)]^2 + [7[2k(k+1)]]^2 = [7\ [2k\ (k+1)+1]\ ]^2 \text{, where } k \in \mathrm{N} \qquad (7)$$

Applying the Principle of Mathematical Induction: to prove equation (7)

For k=1 results $21^2 + 28^2 = 35^2$

Hypothesis:

$$[7(2k+1)]^2 + [7(2k(k+1))]^2 = [7[2k(k+1)+1]]^2$$

Thesis:

$$[7(2k+3)]^2 + [7(2(k+1))(k+2)]^2 = [7(2(k+1)(k+2)+1)]^2$$

$$[7(2k+1)+14]^2 + [7(2k)(k+1)+28(k+1)]^2$$
$$= [7(2k(k+1)+1)+28(k+1)]^2$$

$$[7(2k+1)]^2 + 196(2k+1) + 196 + [7(2k(k+1))]^2 + 784\,k(k+1)^2$$
$$+ 784(k+1)^2$$
$$= [7(2k(k+1)+1)]^2 + 392\,[2k(k+1)+1](k+1)$$
$$+ 784(k+1)^2$$

$$196(2k+1) + 196 = 392(k+1)$$

From the previous results we can arrive at the following theorem:

Theorem: The solutions of the Pythagorean Theorem for positive integers a, b, c, being $a < b < c$, and $c = b + s$, being s other positive integer is satisfied

$$a^2 + b^2 = (b+s)^2 = c^2$$

a) When s is an odd positive integer then the values of the triples (a, b, c) are given by:

$a = (2\alpha+1)(2k+1)$, $b = (2\alpha+1)[2k(k+1)]$ and

$c = (2\alpha+1)[2k(k+1)+1]$,

such that

$$[(2\alpha+1)(2k+1)]^2 + [(2\alpha+1)[2k(k+1)]]^2 = [(2\alpha+1)[2k(k+1)+1]]^2 \quad (8)$$

, where $k \in \mathrm{N}$ *and* $\alpha \in \mathrm{N} \cup \{0\}$

Proof:

Applying the Principle of Mathematical Induction: to prove equation (8)

For α=0 results

$$[1(2k+1)]^2 + [1[2k(k+1)]]^2 = [1[2k(k+1)+1]]^2, \quad (1)$$

For α=1 results

$$[3(2k+1)]^2 + [3[2k(k+1)]]^2 = [3[2k(k+1)+1]]^2, \quad (3)$$

Hypothesis:

$$[(2\alpha+1)(2k+1)]^2 + [(2\alpha+1)(2k(k+1))]^2 = [(2\alpha+1)[2k(k+1)+1]]^2$$

Thesis

$$[(2\alpha+3)(2k+1)]^2 + [(2\alpha+3)(2k(k+1))]^2 = [(2\alpha+3)[2k(k+1)+1]]^2$$

Proof:

$$[(2\alpha+1)(2k+1)+2(k+1)]^2 + [(2\alpha+1)(2k(k+1))+4k(k+1)]^2 =$$

$$[(2\alpha+1)[2k(k+1)+1]+2[2k(k+1)+1]]$$

$$[(2\alpha+1)(2k+1)]^2 + 4(2\alpha+1)(2k+1)^2 + 4(2k+1)^2 + [(2\alpha+1)(2k(k+1))]^2 +$$

$$[(4k)(k+1)]^2 + 8k(2\alpha+1)2k(k+1)^2 =$$

$$[(2\alpha+1)[2k(k+1)+1]]^2 + 4[2k(k+1)+1]^2(2\alpha+1) + 4[2k(k+1)+1]^2$$

,

Then

$$4(2\alpha+1)(2k+1)^2+4(2k+1)^2+16k^2(k+1)^2+16k^2(2\alpha+1)(k+1)^2=$$
$$4(2\alpha+1)4k^2(k+1)^2+4(2\alpha+1)+16k(k+1)(2\alpha+1)+16k^2(k+1)^2+$$
$$16k(k+1)+4$$

$$4(2\alpha+1)4k^2+4(2\alpha+1)+16k(2\alpha+1)+4(2k+1)^2=$$
$$4(2\alpha+1)+16k(k+1)(2\alpha+1)+16k(k+1)+4$$
$$4(2\alpha+1)4k^2+16k(2\alpha+1)+16k^2+4+16k=$$
$$16k^2(2\alpha+1)+16k(2\alpha+1)+16k^2+16k+4$$

b) When s is an even number by:
$a=\alpha[2(k+2)]$, $b=\alpha(k+1)(k+3)$ and
$c=\alpha[(k+1)(k+3)+2]$, such that
$[\alpha[2(k+2)]]^2+[\alpha(k+1)(k+3)]^2=[\alpha[(k+1)(k+3)+2]]^2$,
where $k\in \mathrm{N}$ *and* $\alpha\in \mathrm{N}$ (9)

Proof:

Applying the Principle of Mathematical Induction: to prove equation (9)
For α =1 results
$$[2(k+2)]^2+[(k+1)(k+3)]^2=[(k+1)(k+3)+2]^2 \quad (2)$$
Hypothesis:
$$[\alpha[2(2k+2)]]^2+[\alpha(k+1)(k+3)]^2=[\alpha[(k+1)(k+3)+2]]^2$$

Thesis:
$$[(\alpha+1)[2(2k+2)]]^2+[(\alpha+1)(k+1)(k+3)]^2=[(\alpha+1)[(k+1)(k+3)+2]]^2$$

$$[\alpha[2(k+2)]+2(k+2)]^2+[\alpha(k+1)(k+3)+(k+1)(k+3)]^2=$$
$$[\alpha[(k+1)(k+3)+2]+[(k+1)(k+3)+2]]^2$$

$$\left[\alpha\left[2(k+2)\right]\right]^2+8\alpha(k+2)^2+4(k+2)^2+\left[\alpha(k+1)(k+3)\right]^2+$$
$$2\alpha(k+1)^2(k+3)^2+(k+1)^2(k+3)^2=\left[\alpha\left[(k+1)(k+3)+2\right]\right]^2+$$
$$2\alpha\left[(k+1)(k+3)+2\right]^2+\left[(k+1)(k+3)+2\right]^2$$

$$8\alpha(k+2)^2+4(k+2)^2+2\alpha(k+1)^2(k+3)^2+(k+1)^2(k+3)^2=$$
$$2\alpha(k+1)^2(k+3)^2+8\alpha(k+1)(k+3)+8\alpha+(k+1)^2(k+3)^2+4(k+1)(k+3)+4$$

$$8\alpha(k+2)^2+4(k+2)^2=8\alpha(k+1)(k+3)+8\alpha+4(k+1)(k+3)+4$$

$$8\alpha\left(k^2+4k+4\right)+4\left(k^2+4k+4\right)=8\alpha\left(k^2+4k+3\right)+8\alpha+4\left(k^2+4k+3\right)+4$$

From this theorem result the following Corollaries:
Corollary 1: The integers that satisfy the Pythagorean theorem $a^2+b^2=c^2$ for a, b and c positive integers, such that $a<b<c$ must have the following parity:

a	b	c
odd	even	odd
even	odd	odd
even	even	even

Table 8

Corollary 2: The integers that satisfy the Pythagorean theorem $a^2+b^2=c^2$ for a, b and c positive integers, such that $a<b<c$ never can have the following parity:

a	b	c
odd	odd	odd
odd	even	even
even	odd	even
even	even	odd

Table 9

Let us consider the triangles that have sides and area equals to positive integers [2].

a) For triangles with sides a, b and c positive integers, such that $a < b < c$, where $c-b$ is an odd integer, $a = (2\alpha+1)(2k+1)$, $b = (2\alpha+1)[2k(k+1)]$ and $c = (2\alpha+1)[2k(k+1)+1]$, their areas and perimeters are given by:

$$A = (2\alpha+1)(2k+1)*(2\alpha+1)[2k(k+1)]/2$$
$$= (2\alpha+1)^2(2k+1)k(k+1)$$
$$P = (2\alpha+1)(2k+1)+(2\alpha+1)[2k(k+1)]+(2\alpha+1)[2k(k+1)+1]$$
$$= 2(2\alpha+1)(2k+1)(k+1), \text{ then}$$

$$A = \frac{k(2\alpha+1)}{2} * P \quad , \qquad (10)$$

results that when k is an even number the areas are integer multiple of their respective perimeters.

where $k \in \mathrm{N}$ *and* $\alpha \in \mathrm{N} \cup \{0\}$

b) For triangles with sides a, b and c positive integers, such that $a < b < c$, where $c-b$ is an even integer, $a = \alpha[2(k+2)]$, $b = \alpha(k+1)(k+3)$ and $c = \alpha[(k+1)(k+3)+2]$, their areas and perimeters are given by:

$$A = 2\alpha(k+2)*\alpha(k+1)(k+3)/2$$
$$= \alpha^2(k+1)(k+2)(k+3)$$
$$P = 2\alpha(k+2)+\alpha(k+1)(k+3)+\alpha[(k+1)(k+3)+2]$$
$$= 2\alpha(k+2)(k+3)$$

$$A = \frac{\alpha(k+1)}{2} * P \quad , \qquad (11)$$

results that the areas are integer multiple of their perimeters when $\alpha(k+1)$ is an even number.

where $k \in \mathrm{N}$ *and* $\alpha \in \mathrm{N}$

Theorem: For triangles with sides a, b and c positive integers, such that $a < b < c$,

a) when $c-b$ is an odd integer, the magnitude of their areas are integer multiples of their perimeters when k is an even number.

b) when $c-b$ is an even integer, the magnitude of their areas are integer multiples of their perimeters when $\alpha\left(k+1\right)$ is an even number.

All prime numbers appear as sides of right triangles, with exception of the number 2.

From Table 1 and (1) and from Table 2 and (2) we can summarize that the a, b and c positive integers, such that $a<b<c$, **Primitive Pythagorean Triples** are those for which:

$a=2k+1$, $b=2k\left(k+1\right)$, $c=2k\left(k+1\right)+1$, or

$a=2\left(2k+2\right)$, $b=\left(2k+1\right)\left(2k+3\right)$, $c=\left(2k+1\right)\left(2k+3\right)+2$,

where $k\in \mathrm{N}$ (12)

Conclusion

The Pythagorean triples are obtained under the condition $a<b<c$ in tables 1 to 7. In particular, we have obtained the Primitive Pythagorean triples in table 1 and some part on Table 2. Some parities of the magnitudes of the values a, b, c are allowed and others are forbidden (Tables 8 and 9). If $c-b$ is an odd number the areas are integer proportional to their respective perimeters when k is even. If $c-b$ is an even number the areas are integer proportional to their respective perimeters only when $\alpha\left(k+1\right)$ is an even number. All PPT are obtained through (12), using only one parameter.

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References

Teigen, H. G.; Hadwin, D. W. , "On Generating Pythagorean Triples". The American Mathematical Monthly 78(4), pp. 378-379 (1971).

De la Cruz, J.A.; Goehl Jr., J., "Two Interesting Integer Parameters of Integer-sided Triangles". Forum Geometricorum Volume 17 (2017).