

Measure of Synchronization Time of Small-World Networks Using Dominant Eigenvalue

Menaka Navaratna, Ph.D. ‡

Channa Navaratna, Ph.D. †

Abstract

A circadian pacemaker in mammals, located in the suprachiasmatic nucleus of the hypothalamus (SCN), coordinates daily rhythms of physiology and behavior. This includes both temporal adjustments as well as biological rhythms such as heart rate, sleep-wake cycles, hormone release, cell regeneration, etc. Disruptions to circadian rhythms can have adverse health effects [1], [2]. Circadian rhythm is presumed to be generated via the interactive behavior of a network of coupled oscillators. The exact nature of the connection patterns of these networks are yet to be discovered. Topology of nearest neighbor connections have the drawback of long transient response. Strogatz [3] and others have suggested that a few long distance interconnections may have a dramatic effect in reducing the transient time. These interconnections are thought to occur in a random manner. Here we observe that the transient response can be related to a dominant eigenvalue of the linearized system, and hence it is important to understand the probabilistic relationship between the real part of the dominant eigenvalue and the long distance connections. The probability distribution, which determines the choice of long distance connections, plays an important role in determining the transient response. In some cases, the improvement of the transient response time may be captured using the theory of generalized extreme value distributions.

Introduction

In recent years, researchers have been interested in analyzing the connection topologies of networks such as neural networks, social networks, and many other self-organized systems. It is believed that complex functional behavior of various areas of animal cortex can be understood from the dynamical properties of relevant neurobiology networks. Identical dynamical systems coupled with simple geometrically regular methods and static architecture are the most studied networks. Strogatz [3] did much work on network patterns and came up with the concept of a *small world network*, of which most of the connections are local, and a few long distance connections are sprinkled in a random manner.

A nearest neighbor coupling network has the drawback of long transients. Thus phase shifts may be completed in weeks instead of days. Xiao and Chen [4] have shown that adding a few long distance interconnections dramatically accelerates the transient response. Even if we were to fix the number of long distance connections to some small number, still the search space required to enumerate all possible connections is rather large. Therefore, it is typical to assign long distance connections in a random manner.

One of the problems that has not been solved is how to quantify the effects of long distance connections on the transient response times of a small world network. Simulations [4] have shown time and again, that a few generic long distance connections have the same effect in reducing transients as a dozen or so nearest neighbor connections. Yet no deterministic or probabilistic models exist that can accurately predict this differential effect. It is desirable to devise a statistical model that captures the essence of this relationship. It is well known in linear systems theory that the transient dynamics of a system can be related to a dominant eigenvalue of the system [5].

The SCN system we analyze here shows a nonlinear behavior which is modelled by Achermann and Kunz [6]. Eigenvalues of the linearized model determine the resynchronization time. Let λ_1 represent the eigenvalue with the largest real part (dominant eigenvalue). Phase locking requires that $\text{Re}(\lambda_1) < 0$. Engineers typically take resynchronization time to be $3/\text{Re}(\lambda_1)$. Thus, a critical item for understanding the behavior of the network is an estimate of the real part of λ_1 . It is believed in physiology that each cell interacts mostly with its neighbors, and to a lesser extent with some far away cells. This ambiguity of connections patterns in the SCN makes the task of estimating λ_1 harder, but far more interesting.

If long distance connections are formed in a probabilistic manner, then each of the relevant eigenvalues can be thought of as a random variable; hence the dominant eigenvalue is related to the order statistics of a random vector. Thus, it is important to find details of the probability distribution of the dominant eigenvalue. In an actual biological network, the structure may be far too complex to carry out any useful analysis. Therefore, we assume the network structure is that of a symmetrically coupled oscillator network and perform basics of such an analysis. For illustration, we will consider a network of oscillators arranged in a ring, and interconnections between oscillators are assumed to have a cyclic symmetry. Indeed this type of network has been used to model mammalian circadian rhythms in [6].

In 1982, Kronauer [7] proposed a mathematical model based on the Van der Pol oscillator for simulating the effects of light on the amplitude and phase of the circadian pacemaker in mammals. It identifies light is an external stimulus that shifts the phase of the pacemaker. Since the ambient light intensity oscillates with a precise period of 24 hours, the resulting model consists of a nonlinear oscillator with a 24.3 hour intrinsic period, driven by an external oscillatory input with a 24 hour period. Kronauer's model represents the entire SCN as a two dimensional dynamical system. In 1999, Achermann and Kunz [6] refined Kronauer's model by representing each SCN cell as a Kronauer oscillator, and added coupling terms between oscillators to represent dynamic interactions with other SCN neurons. Their numerical simulations showed that the network is indeed capable of achieving synchrony.

Although the exact interconnectivity neuron is unknown, mathematical modeling of the circadian clock provides framework for understanding available experimental observations and for performing theoretical studies. Here we will discuss the dynamics of a one-dimensional SCN simulation using the Achermann and Kunz model without the effect of ambient light, with the goal of quantifying the effects of random long distance connections on the transient time. Standard Hopf Bifurcation arguments explain how the network will eventually achieve phase locked synchrony (see [8]). This behavior is a direct result of the fact that the network parameters are such that precisely one mode goes through Hopf bifurcation. Therefore, one may conclude that the time elapsed before resynchronization following a reset, e.g. jet lag, is equal to the time for the transient dynamics of other modes to die out.

Here we derive approximate expressions for the dominant eigenvalue in terms of the probability distribution of the long distance connections. We assume that the connection pattern consists of a fixed number of nearest connections and a fixed number of randomly arranged long distance connections. In the case when the number of nearest neighbor connections is reasonably large, it is postulated that the dominant eigenvalue satisfies an extreme value distribution, e.g. a Gumbel, Weibull, or Frechet distribution. Via a simulated neuronal network, we demonstrate how one may fit a statistical distribution to capture the effects of long distance connections on the transient dynamics of the network. We show how one may use the analytical expression for the dominant eigenvalue and Monte-Carlo methods to fit a generalized eigenvalue distribution in order to quantify statistical properties of the dominant eigenvalue.

Oscillators and Eigenvalues

Here we consider a network consisting of N nonlinear oscillators arranged into the form of a ring. Each of the oscillators represents a neuron and is represented by a van der Pol oscillator. The i^{th} neuron is assumed to communicate with a subset of the remaining neurons, and the entire network is assumed to be circular symmetric. The state of the i^{th} neuron is designated by the scalar variables x_i and y_i . Following Acherman and Kunz we represent the network dynamics in the absence of light inputs by,

$$\begin{aligned} \left(\frac{12}{\pi}\right) \dot{x}_i &= y_i + \dot{\alpha} \left(x_i - \frac{4}{3} x_i^3\right) + c \sum_{l=1}^N \theta_l (x_{i+l} - x_i), \\ \left(\frac{12}{\pi}\right) \dot{y}_i &= -\left(\frac{24}{\tau}\right)^2 x_i + c \sum_{l=1}^{N-1} \theta_l (y_{i+l} - y_i), \end{aligned} \tag{1}$$

where $\dot{\alpha}$, c , and $\{\theta_a\}_{a=1}^N$ are positive constants, and $i=1, \dots, N$. Here c is referred to as the coupling strength, and its value is taken to be 0.5. The constant

δ determines the size of the periodic orbit of the Van der Pol oscillator, and its value is taken to be equal to 0.13. Constant θ_l is equal to either 1 or 0 depending upon whether or not a connection of length l exists. For the sake of simplicity, we take $\tau = 24$.

Standard computations (see e.g. [9]) can be carried out to show that the eigenvalues of the linearization of (1) are:

$$\lambda_0 = (\pi/12)(\delta/2 \pm i\sqrt{1-\delta^2/4}),$$

$$\lambda_k = \lambda_0 + (c\pi/12) \sum_{l=1}^{N-1} (e^{i\frac{2\pi lk}{N}} - 1)\theta_l, \quad (2)$$

$$k = 1, \dots, N-1,$$

and the corresponding eigenvectors are $V_k = [v_{k,1}, v_{k,2}, \dots, v_{k,N}]^T$, where $v_{0,l}$ is the eigenvector of an individual oscillator with no connections corresponding to its eigenvalue λ_0 , and

$$v_{k,l} = v_{0,l} e^{i\frac{2\pi lk}{N}} \quad (3)$$

Thus, phase locked oscillations demand that $Re(\lambda_0) > 0 > \lambda_k$ for $k = 1 \dots N-1$. Moreover, transient response due to a resynchronization of the SCN clocks will be determined by time constants corresponding to the $2(N-1)$ stable eigenvalues. If we were to adopt the engineers' rule of thumb that the time it takes for an exponential to die out is equal to three times the time constant, then the transient response time can be estimated as,

$$T_{\text{transient}} = \frac{3}{\min_{k=1}^{N-1} (Re\lambda_k)}, \quad (4)$$

In the particular case when each cell is connected to its m nearest neighbors, we may write simplified expressions for the eigenvalues,

$$\lambda_k = \lambda_0 - (c\pi/12) \left[(2m+1) - \frac{\sin\left(\frac{(2m+1)\pi k}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)} \right]. \quad (5)$$

This expression demonstrates the main problem associated with the nearest neighbor only type of connections.

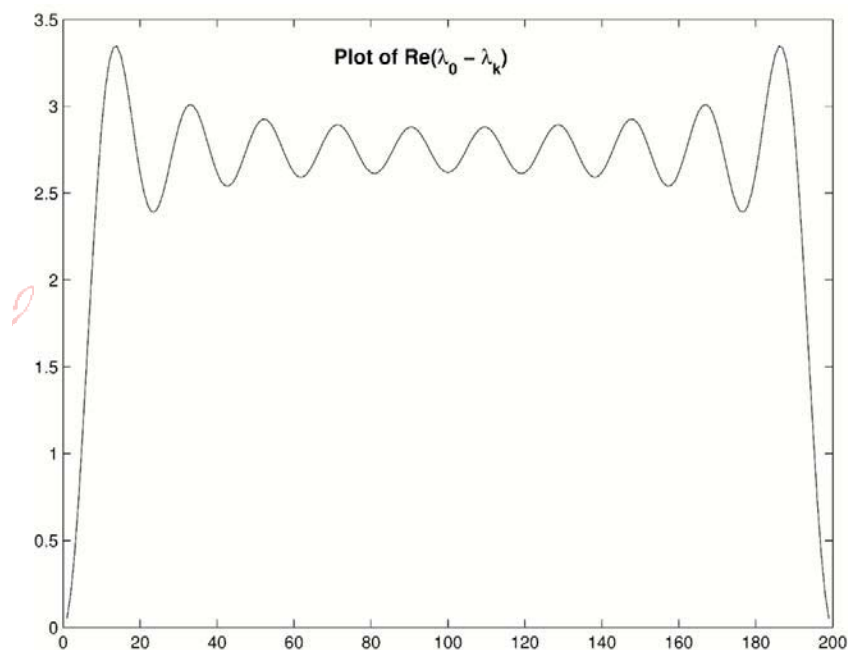


Figure 1

k vs $\lambda_0 - \lambda_k$ plot for a network with 200 cells. Each cell is coupled to 10 nearest neighbors on either side.

A typical plot of $\lambda_0 - \lambda_k$ is shown in Figure 1 in which there are 200 cells in a ring and each cell communicates with its ten nearest neighbors on either side. It is observed that most of the eigenvalues have fairly large negative real parts, whereas one or two of the eigenvalues have a very small negative real part, thereby causing a relatively slow decay of the transient dynamics. If we were to add just three long distance connections, one of length 100, and two of length 50 on either side, then the expression for $\lambda_0 - \lambda_k$ will have an extra additive term of $(c\pi/12)(2\cos(\pi k/2) + (-1)^k - 3)$, which pushes the dominant eigenvalue from the previous position of -0.05 to the left by 0.68.

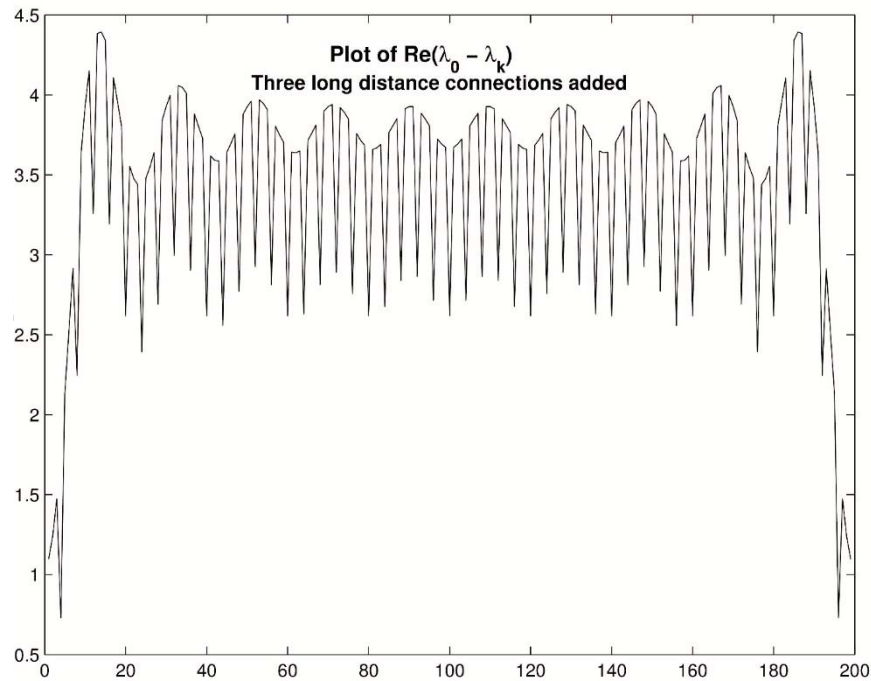


Figure 2

k vs $\lambda_0 - \lambda_k$ plot for a network with 200 cells. Each cell is coupled to its 10 nearest neighbors on either side, to cells 50 units on either side, and to the cell 100 units away.

The new plot with three long distance connections is shown in Figure 2.

Probability Density of Eigenvalues

As seen in the example above, a few random long distance connections are generally sufficient to move the dominant eigenvalue to the left by a significant amount in a probabilistic sense [4].

We have already observed that the transient response of the network considered here, i.e. a ring model of the mammalian circadian rhythm generator, can be directly linked to the dominant stable eigenvalue of its linearized dynamics. Let us denote its real part by α_{dom} . From what we have seen above, α_{dom} is equal to,

$$\alpha_{\text{dom}} = \max_{k=1}^{N-1} \text{Re} \left(\lambda_0 + (c\pi / 12) \sum_{l=1}^{N-1} \left(e^{i \frac{2\pi lk}{N}} - 1 \right) \theta_l \right). \quad (6)$$

Let us assume that a certain number of nearest neighbor connections always exist, i.e. $\theta_k = 1$; $k = \pm 1, \dots, \pm m$. In addition, we assume that a certain fixed number, $2p$, of long distance connections exist, and the lengths of them are picked at random according to an a priori defined probability density function. The set A denotes the set of indices of the corresponding long distance connections. In order to keep the notation and analysis simple, we allow for the possibility that A may contain small indices as well. Considering the fact that the number of oscillators is typically very large in comparison to p , the probability that a small index is included in A is indeed very small.

Let us assume that long distance connections are chosen independently of each other and in accordance with a given probability density function f^A , subject to the constraint that A satisfy \pm symmetry, i.e. $j \in A$ implies that $-j \in A$. Let X_k denote the random variable $2\cos(2\pi lk / N)$ where l is distributed according to f^A . Let Y_k be the sum of p independent random variables X_k . It now follows from (6) that the real part of the dominant eigenvalue α_{dom} is the random variable,

$$\alpha_{\text{dom}} = \max_{k=1}^{N-1} \text{Re}(\lambda_k) + (c\pi / 12)(Y_k - 2p). \quad (7)$$

where λ_k is given in (5). Thus we observe that the random variables $\{Y_k\}_{k=1}^N$ capture the probabilistic aspects of the dominant eigenvalue entirely. As noted earlier, index $l \in [1, N]$ is picked according to the density f^A . If f^k were to denote the probability density of $X_k = 2\cos(2\pi lk / N)$, then Y_k in (7) has the density,

$$f^{Y_k}(y) = (f^k * f^k * \dots * f^k)(y), \quad (8)$$

where the right hand side consists of the p fold convolution operation. Since X_k has support in $[-2, 2]$, it follows that Y_k has support in $[-2p, 2p]$, and from (7) we observe that the probabilistic correction term of the dominant eigenvalue always has a stabilizing effect, and has the support in $[-pc\pi / 6, 0]$.

Case I: very small number of long distance connections

Calculation of $f^{Y_k}(y)$ in (8) is too complicated to be of much analytical usefulness. However, if one studies the small world network problem from the viewpoint of choosing an appropriate probability density f^A , then a good

starting point is to ensure that the probability that Y_k lies in a certain small neighborhood $[2p - \delta, 2p]$ is reasonably small (say δ). This way the probabilistic terms will push the dominant eigenvalue to the left by $c\pi\delta/12$ with a reasonably high probability $1 - \delta$.

If the number of long-distance connections are comparably small to the number of local connections, then it is reasonable to assume that the conditional probability that $Re(\lambda_k) + (c\pi/12)(Y_k - 2p) > Re(\lambda_1) + (c\pi/12)(Y_1 - 2p)$ for $k > 1$ in (7) conditioned on the event that the probabilistic effects of p on α_{dom} is small. This effect is supported by the significant difference between $Re(\lambda_1)$ and $Re(\lambda_k)$ as seen in Figure 1. There we approximate,

$$\alpha_{dom} = Re(\lambda_1) + (c\pi/12)(Y_1 - 2p) \quad (9)$$

The advantage of (9) is that one may often obtain analytical expressions for the density of Y_1 in the right end of its support, which is all we are interested in.

This is illustrated in the following example.

Example 1: Long distance connections are picked according to the uniform density function.

We assume that N is very large so that we may approximate $2\pi l/N$ by a continuous uniform random variable $\Theta \in [0, 2\pi]$. Therefore, $X_1 = 2\cos(\Theta)$ has density,

$$f^1(x) = \frac{1}{2\pi\sqrt{1-x^2/4}}, \quad x \in [-2, 2]. \quad (10)$$

Since we only desire to compute the density of Y_1 at values in the vicinity of $2p$, we need values of $f^1(x)$ for x near 2 , which may be approximated by,

$$f^1(2-z) \approx \frac{1}{2\pi\sqrt{z}}. \quad (11)$$

Using (11) we may approximate the p fold convolution of f^1 to obtain the density of Y_1 for small positive values of y as,

$$f^{Y_1}(2p-y) \approx c_p y^{\frac{p-1}{2}}, \quad (12)$$

where c_p are constants.

The uniform probability law is not very appropriate for choosing long distance connections as it may lead to selection of long distance connections that are actually nearest neighbor connections. In view of the example illustrated in Figure 2, one may argue that a probability law f^A , which will allow more connections toward the middle cells, i.e. connections of length close to $N/2$, may have more of a stabilizing effect. This hypothesis is tested in Example 2.

Example 2: Long distance connections are picked according to the density,

$$f^\Theta(\theta) = |\sin(\theta)|/4. \text{ Now, } X_1 = 2\cos(\Theta) \text{ has density,} \\ f^1(x) = 1/4, x \in [-2, 2]. \quad (13)$$

Thus, we may approximate the p fold convolution of f^1 to obtain the density of Y_1 for small positive values of y as,

$$f^{Y_1}(2p - y) \approx c_p y^{p-1}, \quad (14)$$

where c_p are constants. By comparing exponents, it is seen that the density in (14) is far smaller than the density in (12) for small values of y , hence the probability that long distance connections may fail to have a significant stabilizing effect is far smaller in Example 2 than in Example 1.

Case II: relatively large number of long distance connections

For large number of long distance connections, it safe to assume that the probability of α_{dom} may correspond to some Y_k for $k > 1$. However, α_{dom} being the largest of a set of random variables, it can be conjectured that it is probabilistically distributed according to an extreme value distribution. Thus, we will be able to adopt the theory of generalized extreme value distributions. It has been shown that under certain mild constraints, extreme values of random processes fall into one of three distributions, *Gumble*, *Frechet* or *Weibull*. The key parameters which govern the behavior of these distributions are μ the *location*, σ the *scale*, and ξ the *shape*. The definition for generalized extreme value distribution function are,

$$\text{Gumbel: } G(z) = \exp \left\{ -\exp \left[-\left(\frac{z-b}{a} \right) \right] \right\}, z \in \mathfrak{R},$$

$$\text{Frechet: } G(z) = \exp \left\{ \begin{array}{ll} 0, & z \leq b \\ \exp \left[-\left(\frac{z-b}{a} \right)^{-\alpha} \right], & z > b \end{array} \right\}$$

$$\text{Weibull: } G(z) = \exp \left\{ \begin{array}{ll} \exp \left[- \left(\frac{z-b}{a} \right)^\alpha \right] & z < b \\ 1 & z \geq b \end{array} \right\}.$$

The three distributions can be combined into a single family of models having a distribution function,

$$G(z) = \exp \left\{ - \left[1 + \xi \left(\frac{z - \mu}{\sigma} \right) \right]^{-1/\xi} \right\}. \quad (15)$$

This family is referred to as the Generalized Extreme Value distributions, or GEV for short. One may carry out a parameter estimation to fit a given set of data using standard procedures such as maximum likelihood estimation.

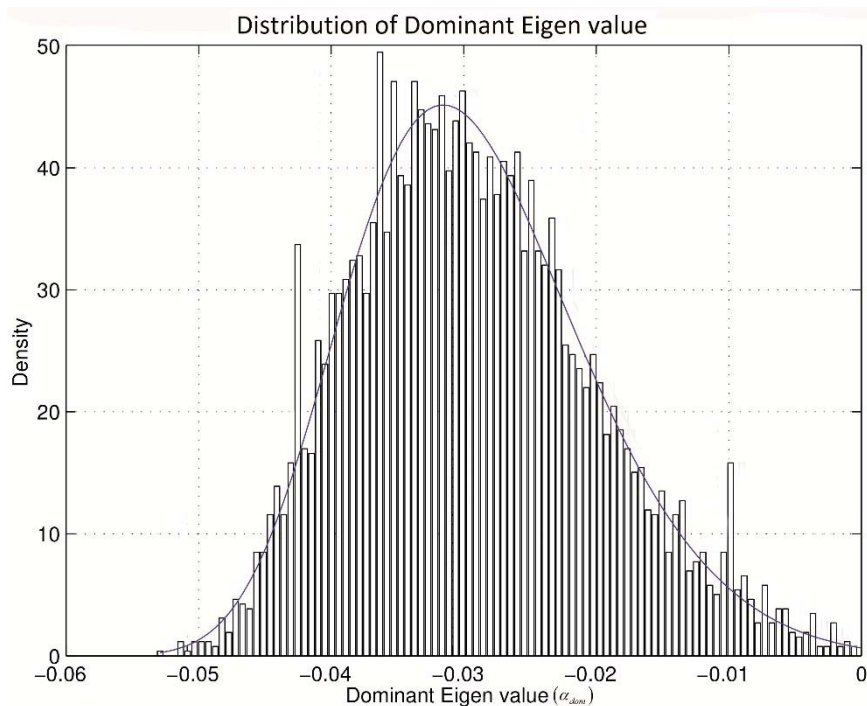


Figure 3

Generalized Extreme Value distribution of α_{dom} in a network with 15 random connections of length between 0 and 15 and 10 random connections of length between 25 and 475 in a ring of 500 oscillators. Estimated parameter values are: $\mu = -0.0325$, $\sigma = 0.0081$, $\xi = -0.1532$

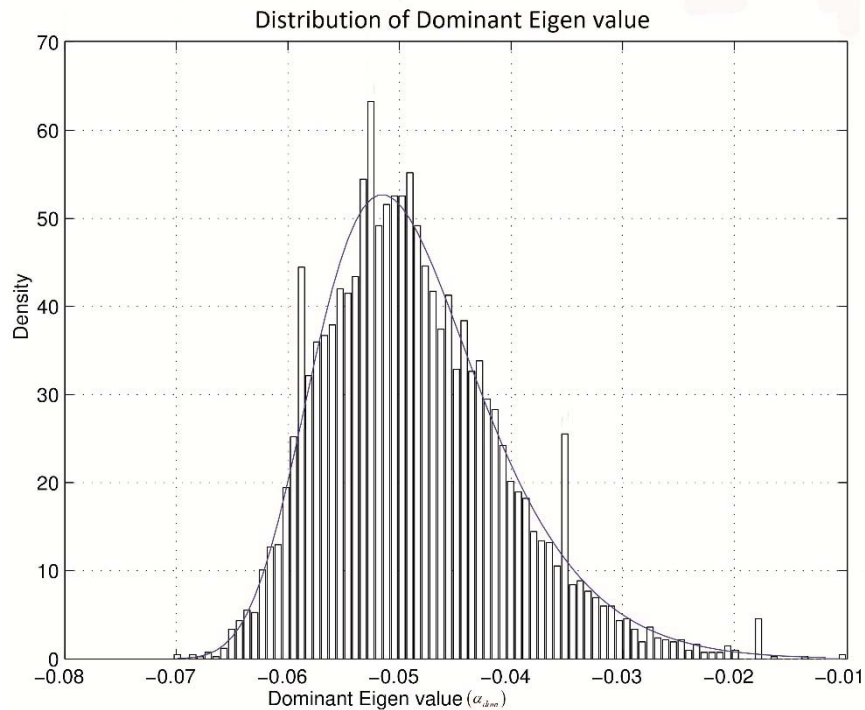


Figure 4

Generalized Extreme Value distribution of α_{dom} in a network with 25 random connections in a ring of 500 oscillators. $\mu = -0.0501$, $\sigma = 0.0072$, $\xi = -0.0761$.

We have used repeated trials to compute sample values of α_{dom} in numerical experiments. The maximum likelihood method was used to estimate the relevant parameters of a GEV distribution. Figures 3 and 4 show histograms of numerical values of α_{dom} and their GEV fits. The network in Figure 3 consists of a SCN model (1) in which 10 nearest neighbor connections were placed at random, and their lengths were restricted to be at most 15. In addition, ten long distance connections were chosen at random under the restriction that their lengths must exceed 25. In the network in Figure 4, the same number of oscillators were used, but connections were limited to 30, and they were placed completely at random.

Concluding Remarks

We have given a conjecture for the distribution of highest eigenvalue and its distribution under various network types. We have described a means to capture probabilistic aspects of resynchronization time in a small world network via the

example of a mathematical model of the circadian rhythm generator circuitry. Two cases are considered; one in which the number of long distance connections are very small, and another in which this number is reasonably large in comparison to nearest neighbor type connections. An analytical mechanism is described to find an approximation to the probability density of the dominant eigenvalue in the first case. In the second case, it is argued that the Generalized Extreme Value distributions are a reasonable means to describe the dominant eigenvalues of such a network. Some numerical evidence is provided to support the thesis. Work is under way to provide a rigorous proof of this idea.

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‡ Menaka Navaratna, Ph.D., Florida Gulf Coast University, Florida, USA

† Channa Navaratna, Ph.D., Indiana University of PA, Pennsylvania, USA

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