Application of Grey Numbers to Student Assessment Under Fuzzy Conditions

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Abstract

The grey numbers are defined with the help of the real intervals and they play an important role in the everyday life for handling approximate data. In the present paper grey numbers are used as a tool for estimating the mean performance of a student class when the individual student evaluation is made with linguistic grades, which involve a degree of fuzziness. A classroom application is also presented illustrating our results. Although the proposed new assessment method is proved to be equivalent with an analogous method using Triangular Fuzzy Numbers developed in earlier works, the required computational burden is significantly reduced.

Keywords: Grey Numbers (GNs), Whitenization, Student Assessment

1. Introduction

The student assessment is a very important task of Education, because apart of being a social need and demand it helps the instructors in designing their future plans for a more effective teaching procedure. When the student performance is evaluated with numerical scores, then the traditional way to assess the *mean performance* of a student class is the calculation of the average of those scores. However, either for reasons of more elasticity or to comfort the teacher's existing uncertainty about the exact value of the numerical scores corresponding to each student's performance, frequently in practice the assessment is made not by numerical scores but by linguistic grades, like excellent, very good, good, etc. This involves a degree of fuzziness and makes the calculation of the mean value of grades impossible.

A popular in such cases method for evaluating the overall performance of a student class is the calculation of the *Grade Point Average (GPA) index*. For this, let n be the total number of students of the class and let n_A, n_B, n_C, n_D and n_F be the numbers of students who demonstrated excellent (A), very good (B), good (C), fair (D) and unsatisfactory (F) performance respectively. Then the GPA index is calculated by the formula $GPA = \frac{0n_f + 1n_D + 2n_C + 3n_B + 4n_A}{n}$ GPA = (e.g. see [4], Chapter 6, p.125). Therefore GPA is a weighted average in which greater coefficients (weights) are assigned to the higher grades, which means that it reflects not the mean, as we wish, but the *quality performance* of the student class.

In earlier works and in an effort to estimate the mean student performance in such fuzzy assessment cases, we have used tools from *Fuzzy Logic*. More explicitly, representing the student class as a fuzzy set in the set U = {A, B, C, D, F} of the above described linguistic grades, we calculated the existing in it *uncertainty*, based on the classical principle that the greater is the reduction of the uncertainty the more is the new information obtained by the class and therefore the better is the student performance (e.g. see Section 3 of [3], or Chapter 5 of [4]). However, this method has two disadvantages: First it involves laborious calculations and second it can be used for comparing the performance of two different classes only under the assumption that they have been proved to be equivalent before the corresponding activity (e.g. test, problem-solving, learning a new subject matter, etc.), a condition that does not hold always in practice. For this reason we have used later *Triangular Fuzzy Numbers (TFNs)* for assessing the student mean performance (see [6] or Chapter 7 of [4]), a method that has been proved to be easier and more accurate than the calculation of the uncertainty.

The purpose of the present paper is to develop an alternative method of estimating the student mean performance that uses *Grey Numbers (GNs)* instead of TFNs. Although the two methods will be proved to be equivalent to each other, the use of the GNs reduces significantly the required computational burden. The rest of the paper is formulated as follows: In Section 2 we introduce the necessary for the understanding of the present paper background from the theory of GNs. In Section 3 we develop our new assessment method, while in Section 4 we present a classroom application illustrating our results. Finally, our conclusion is stated in Section 5 together with some hints for future research.

2. Grey Numbers

Frequently in the everyday life, as well as in many applications of science and engineering including medicine diagnostics, psychology, sociology, control systems, economy price indices, opinion polls, etc., the data cannot be easily determined precisely and in practice estimates of them are used. Apart from fuzzy logic, another effective tool for handling the approximate data is the use of the GNs, which are introduced with the help of the real intervals.

A GN is an indeterminate number whose probable range is known, but which has unknown position within its boundaries. Therefore, if $\boldsymbol{R}$: denotes the set of real numbers, a GN, say A, can be expressed mathematically by

$$A \in [a, b] = \{x \in \boldsymbol{R} : a \leq x \leq b\}.$$

Compared with the interval [a, b] the GN A enriches its uncertainty representation with the *whitenization function*, defining a *degree of greyness* for each x in $[a, b]$. If $a=b$, then A is called a *white number* and if $A \in (-\infty, +\infty)$, then A is called a *black number*. For general facts on GNs we refer to the book [1].

From the definition of the GNs it becomes evident that the well known arithmetic of the real intervals [2] can be used to define the basic arithmetic operations among the GNs. More explicitly, if $A \in [a_1, a_2]$ and $B \in [b_1, b_2]$ are two given GNs, then we define:

- *Addition* by: $A + B \in [a_1 + b_1, a_2 + b_2]$
- *Subtraction* by: $A - B = A + (-B) \in [a_1 - b_2, a_2 - b_1]$, where $-B \in [-b_2, -b_1]$.

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Observe that B + (-B) $\in [b_1 - b_2, b_2 - b_1] \neq [0, 0] = 0$ and also B + (-B) $\neq$ (-B) + B $\neq$ 0.

- *Multiplication* by: A x B $\in [min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}]$
- *Division* by: A : B = A x B$^{-1} \in [\min\{\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\} \max\{\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\}]$

, where $0 \notin [b_1, b_2]$ and B$^{-1} \in [\frac{1}{b_2}, \frac{1}{b_1}]$

Observe that B x B^{-1} = B^{-1} x B $\in [\frac{b_1}{b_2}, \frac{b_2}{b_1}] \neq [1, 1] = 1$

- *Scalar multiplication* by: kA $\in [ka_1, ka_2]$, where k is a positive real number.

Let us denote by w(A) the white number with the highest probability to be the representative real value of the GN A $\in [a, b]$. The technique of determining the value of w(A) is called *whitenization* of A. One usually defines w(A) = $(1- t)a + tb$, with t in [0, 1]. This is known as *equal weight whitenization*. When the distribution of A is unknown, we take $t = \frac{1}{2}$, which gives that $w(A) = \frac{a+b}{2}$.

3. The Assessment Method with the GNs

Let G be a group of n students participating in a certain activity (e.g. learning a new subject matter, problem-solving, etc.). Assume that one wants to estimate the mean performance of G in terms of the linguistic grades A, B, C, D and F presented in Section 1. For this, we introduce a numerical scale of scores from 0 to 100 and we correspond these scores to the linguistic grades as follows: A (100-85), B(84-75), C (74-60), D(59-50) and F(49-0). This correspondence, although it satisfies the common sense, it is not unique, depending on the observer's personal goals. For example, in a more strict assessment one could take A (100-90), B(89-80), C (79-70), D(69-60) and F(59-0), etc.

It is possible now to represent each linguistic grade by a GN, denoted for simplicity with the same letter. Namely, we introduce the GNs: $A\in[85, 100]$, $B\in[75, 84]$, $C\in[60, 74]$, $D\in[50, 59]$ and $F\in[0, 49]$. Let n_A, n_B, n_C, n_D and n_F be the numbers of the students of G whose performance was evaluated by the grades A, B, C, D and F respectively. Assigning to each student the corresponding GN we define the *mean value* of all those GNs to be the GN:

$$M = \frac{1}{n}[n_A A + n_B B + n_C C + n_D D + n_f F]$$

and we take M as a representative of the group's mean performance.

In fact, $n_A A\in[85n_A, 100n_A]$, $n_B B\in[75n_B, 84n_B]$, $n_C C\in[60n_C, 74n_C]$,
$n_D D\in[50n_D, 59n_D]$ and $n_F F\in[0n_F, 49n_F]$. Therefore $M\in[m_1, m_2]$, with

$m_1=\frac{85n_A+75n_B+60n_C+50n_D+0n_f}{n}$ and $m_2=\frac{100n_A+84n_B+74n_C+59n_D+49n_f}{n}$.

Consider now the extreme case where the maximal possible numerical score corresponds to each student for each linguistic grade, i.e. the n_A scores corresponding to A are 100, the n_B scores corresponding to B are 84, etc. In this case the mean value of all those scores is equal to m_2. Also, for the other extreme case where the minimal possible numerical score corresponds to each student for each linguistic grade, i.e. the n_A scores corresponding to A are 85, the n_B scores corresponding to B are 75, etc. the mean value of all those scores is equal to m_1. Since the distributions of the GNs A, B, C, D and F are unknown, the same happens with the distribution of M. Therefore we can take

$$w(M) = \frac{m_1+m_2}{2}.$$

Consequently, the assessment method with the GNs gives a good approximation of the group's mean performance and therefore it is useful when no numerical scores are used and the group's performance is assessed by qualitative grades (see also [5]).

Remark: In earlier works (see [6] or Chapter 7 of [4]) he have estimated the mean performance of G by using the TFNs A = (85, 92.5, 100}, B = (75, 79.5, 84), C = (60, 67, 74], D = (50, 54.5, 59] and F = (0, 24.5 49] instead of the corresponding GNs used here.

Assigning to each student the TFN corresponding to his/her performance we calculated the mean value of all those TFNs and by defuzzifying it with the *Center of Gravity (COG) technique* we have found the same value $\frac{m_1+m_2}{2}$.

Consequently the two assessment methods (TFNs and GNs) are equivalent to each other, but the method with the GNs reduces significantly the computational burden.

4. A Classroom Application

The following Table depicts the performance of two student groups, say G_1 and G_2, in a common mathematical examination. Compare the mean performance of the two groups.

Table 1: Student performance

Grade	G_1	G_2
A	20	20
B	15	30
C	7	15
D	10	10
F	8	10
Total	60	85

Assigning to each student the corresponding GN we calculate the mean values M_1 and M_2 of all those GNs for the groups G_1 and G2 respectively, which are approximately equal to:

$M_1 = \frac{20A+15B+7C+10D+8F}{60} \in [62.42, 79.33]$

$M_2 = \frac{28A+30B+15C+10D+10F}{85} \in [62.94, 78.94]$

Therefore

$w(M_1) = \frac{62.42+79.33}{2} = 70.88$ and $w(M_2) = \frac{62.94+78.94}{2} = 70.94$

Consequently both groups demonstrated a good (C) mean performance, with the mean performance of the second group being slightly better.

Remark: Calculating the GPA index for the two groups one finds that

$$GPA = \frac{4.20 + 3.15 + 2.7 + 1.10}{60} = 2.48$$

for the first and

$$GPA = \frac{4.20 + 3.30 + 2.15 + 1.10}{60} = 2.47$$

for the second group. Therefore, in contrast to the mean performance, the first group demonstrated a slightly better quality performance than the second one.

5. Conclusion

The traditional method of assessing the mean performance of a student group by calculating the average of the student numerical scores cannot be applied when the student performance is evaluated by qualitative grades. Also, the calculation of the GPA index that can be applied in such cases measures the group's quality performance by assigning greater coefficients to the higher grades. For this reason we have developed two methods for estimating the group's mean performance under fuzzy conditions. The first one, developed in earlier works, uses a combination of TFNs and the COG defuzzification technique for this purpose. Although the second method with the help of GNs that has been developed in the present paper was proved to be equivalent with the first one, the required computational burden was significantly reduced.

Grey numbers play in general an important role in science, engineering and in the everyday life for handling approximate data and the development of further applications of them to real life problems will be one of the main components of our future research.

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