Geometric Relationships of Power Functions

Richard Winton, Ph.D. †
Sarah S. Horner, PE ‡

Abstract

Several regions are defined in the Cartesian plane which are related to power functions of the form f(x) = x^r $(0 \le x \le t)$, where r and t are positive real numbers. Relationships between the areas of these regions are explored as functions of r and t. Additional properties of these areas and their relationships are investigated in the extreme cases in which r approaches zero and r approaches infinity.

Introduction

Basic power functions and those functions related to them possess a number of interesting properties and symmetries. Furthermore, exploring these characteristics of power functions can provide good undergraduate research projects for calculus students in a limited time frame.

For example, in 2005 Richard A. Winton derived the coordinates of the centroid C_n of the region R_n in the Cartesian plane bounded by the power function f(x) = x^n (where n is an integer; n > 1; $0 \le x \le 1$) and its inverse function $f^{-1}(x) = \sqrt[n]{x} = x^{1/n}$ $(0 \le x \le 1)$ as a rational function of the exponent n [1, p. 230]. In particular, Winton showed that for each integer n > 1, C_n = $\left(\frac{n^2+2n+1}{2n^2+5n+2}, \frac{n^2+2n+1}{2n^2+5n+2}\right)$. As well as an obvious symmetry in R_n about the line y = x due to R_n being bounded by a function and its inverse, the question of an another symmetry in R_n about the line y = 1–x was raised. Such an additional symmetry, if it existed, would force the centroid C_n to be $\left(\frac{1}{2}, \frac{1}{2}\right)$. Although $C_n \neq \left(\frac{1}{2}, \frac{1}{2}\right)$ for each integer n > 1, the symmetry of R_n relative to the line y = 1–x was established for the limiting case of C_∞ in which n approaches infinity [1, p. 231]. That is,

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty}\left(\frac{n^2+2n+1}{2n^2+5n+2}, \frac{n^2+2n+1}{2n^2+5n+2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

As another example, in 2010 Winton and Warren developed a generalized formula for the surface area generated by revolving a continuous curve which is defined over a closed, bounded interval about an arbitrary linear axis of revolution [2, p. 32]. More specifically, if the graph of a continuous function y = f(x) ($a \le x \le b$) is revolved about an arbitrary linear axis L, then the resulting surface area generated is

$$SA = \begin{cases} 2\pi \int_a^b |x-t|\sqrt{1+[f'(x)]^2}\,dx & \text{if L is vertical defined by } x = t \\ \dfrac{2\pi}{\sqrt{m^2+1}} \int_a^b |f(x)-L(x)|\sqrt{1+[f'(x)]^2}\,dx & \text{if } L(x) = mx + t. \end{cases}$$

Thus in the specific case in which a power function $f(x) = x^r$ (where r is a real number; $r > 0$; $a \le x \le b$) is revolved about a linear axis L, the surface area generated is

$$SA = \begin{cases} 2\pi \int_a^b |x-t|\sqrt{1+r^2x^{2r-2}}\,dx & \text{if L is vertical defined by } x = t \\ \dfrac{2\pi}{\sqrt{m^2+1}} \int_a^b |x^r-L(x)|\sqrt{1+r^2x^{2r-2}}\,dx & \text{if } L(x) = mx + t \end{cases}$$

We now proceed with the primary goal of this paper. The first step will be to establish the specific regions in question which will be the focus of the investigation and their respective areas in the Cartesian plane $\mathbf{R}^2$.

Basic Definitions

Throughout this paper r and t will denote positive real numbers, so that $f(x) = x^r \ge 0$ for $0 \le x \le t$. In $\mathbf{R}^2$, define R(r,t) to be the rectangle with horizontal and vertical sides circumscribed about the graph of $f(x) = x^r$ ($0 \le x \le t$) at the endpoints (0,0) and (t, t^r). We further define A(r,t) to be the area between the graph of $f(x) = x^r$ ($0 \le x \le t$) and the x-axis, as shown in Figure 1 below. Consequently $A(r,t) = \int_0^t f(x)\,dx = \int_0^t x^r\,dx = \frac{1}{r+1}x^{r+1}\Big|_0^t = \frac{1}{r+1}t^{r+1}$, and so

$$A(r,t) = \frac{1}{r+1}t^{r+1}. \tag{1}$$

In contrast to A(r,t), we define B(r,t) to be the area between the graph of $y = t^r$ and the graph of $f(x) = x^r$ ($0 \le x \le t$), as shown in Figure 2 below.

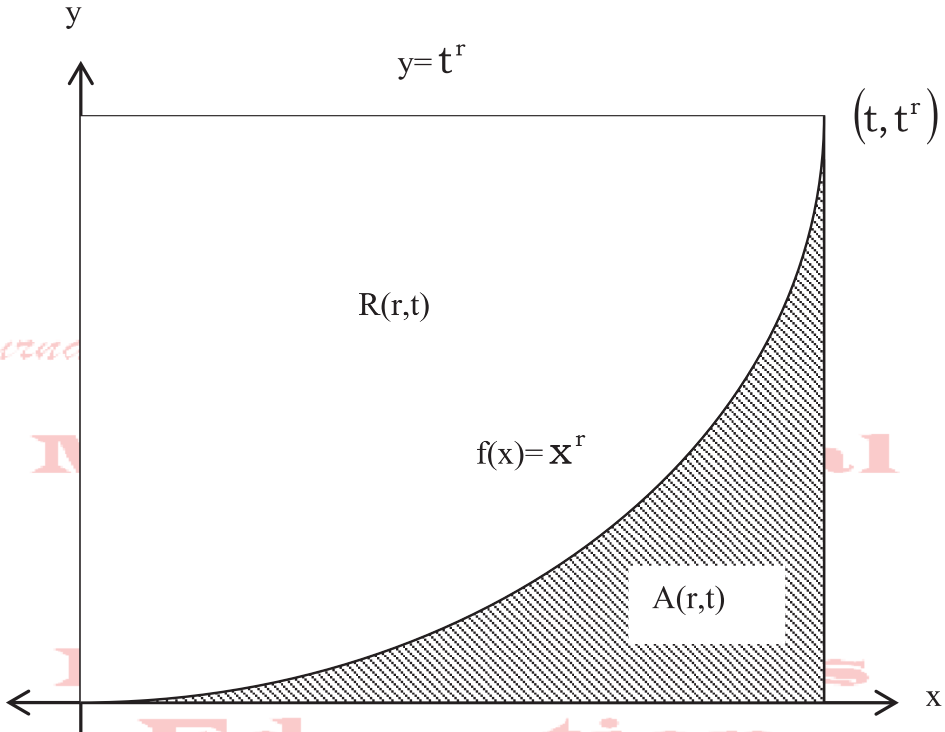

Figure 1

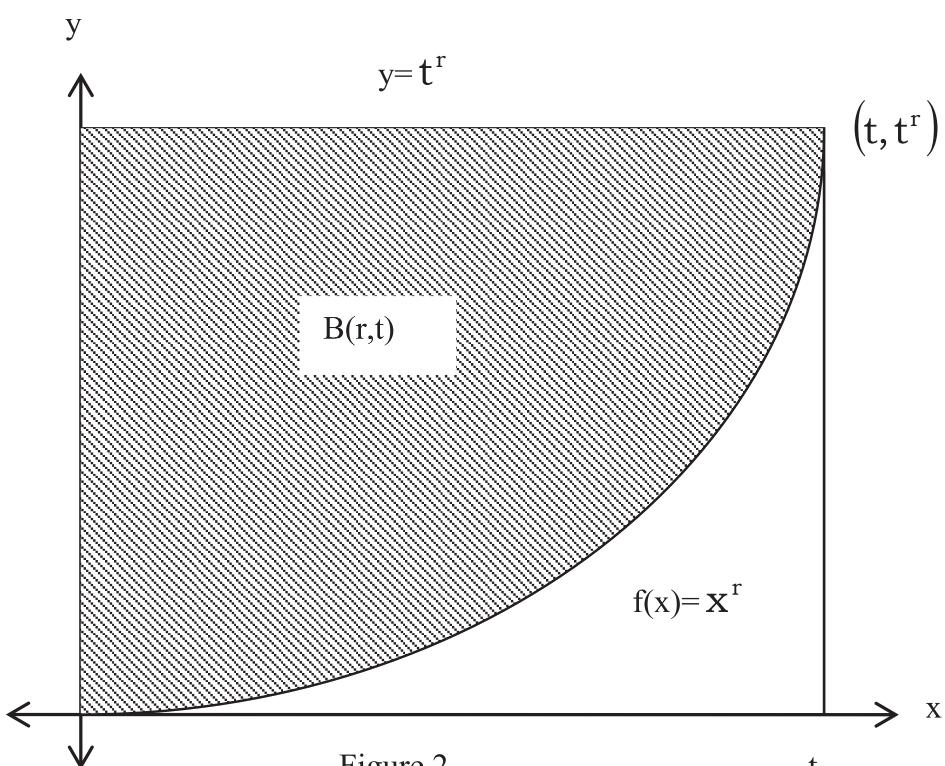

Figure 2

Thus $B(r,t) = \int_0^t \left[t^r - f(x)\right]dx = \int_0^t \left[t^r - x^r\right]dx = t^r x - \frac{1}{r+1}x^{r+1}\Big|_0^t = \frac{r}{r+1}t^{r+1}$, so

$$B(r,t) = \frac{r}{r+1}t^{r+1}. \tag{2}$$

Alternatively, $B(r,t) = \int_{y=0}^{y=t^r} y^{1/r}dy = \frac{1}{\frac{1}{r}+1}\cdot y^{\frac{1}{r}+1}\Big|_{y=0}^{y=t^r} = \frac{r}{r+1}\cdot y^{\frac{r+1}{r}}\Big|_{y=0}^{y=t^r} = \frac{r}{r+1}\left(t^r\right)^{\frac{r+1}{r}}$

$= \frac{r}{r+1}t^{r+1}$, which agrees with (2) above.

Finally, define $C(r,t)$ to be the area between the graph of $y = t^r$ $(0 \le x \le t)$ and the x-axis, as shown in Figure 3 below.

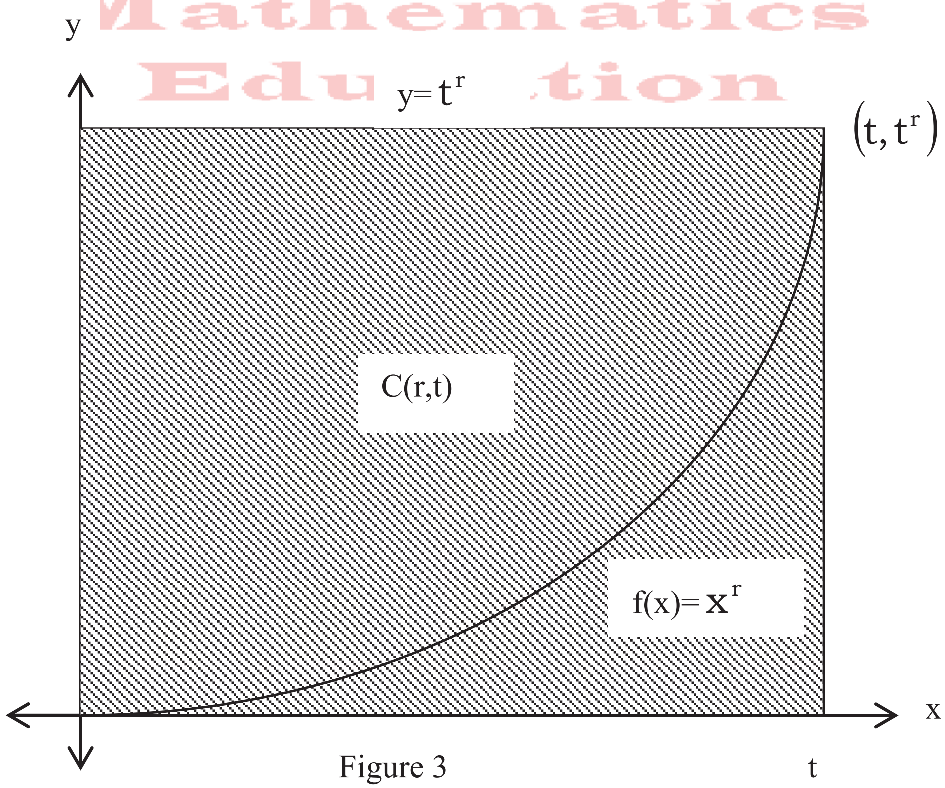

Figure 3

Hence, $C(r,t)$ is the area of the rectangle $R(r,t)$ defined above which is circumscribed about the graph of $f(x) = x^r$ $(0 \le x \le t)$ at its endpoints $(0,0)$ and (t, t^r). Therefore we have

$$C(r,t) = t \cdot t^r = t^{r+1}. \tag{3}$$

It is clear from Figures 1-3 that C(r,t) = A(r,t) + B(r,t). This rather obvious relationship is formally established by the fact that from (1), (2), and (3) we have $A(r,t) + B(r,t) = \frac{1}{r+1}t^{r+1} + \frac{r}{r+1}t^{r+1} = \frac{1+r}{r+1}t^{r+1} = t^{r+1} = C(r,t)$, and so

$$A(r,t) + B(r,t) = C(r,t). \tag{4}$$

Relative Areas

Comparing the relative areas defined by A(r,t), B(r,t), and C(r,t), we have the following results. Firstly, by (1) and (3), we have $\frac{C(r,t)}{A(r,t)} = \frac{t^{r+1}}{\frac{1}{r+1}t^{r+1}} =$ r + 1, so that

$$\frac{C(r,t)}{A(r,t)} = r + 1, \tag{5}$$

which is independent of t. Secondly, (2) and (3) imply that $\frac{C(r,t)}{B(r,t)} = \frac{t^{r+1}}{\frac{r}{r+1}t^{r+1}} =$ $\frac{r+1}{r}$, so that

$$\frac{C(r,t)}{B(r,t)} = \frac{r+1}{r}, \tag{6}$$

which is also independent of t. Finally, and perhaps most surprising of all, applying (1) and (2) produces $\frac{B(r,t)}{A(r,t)} = \frac{\frac{r}{r+1}t^{r+1}}{\frac{1}{r+1}t^{r+1}} = \frac{r}{r+1} \cdot (r+1) = r$, and so

$$\frac{B(r,t)}{A(r,t)} = r, \tag{7}$$

which again is independent of t.

Consequently, in all three cases the ratios of these areas are independent of the parameter t which establishes the interval over which they are defined. Furthermore, within the rectangle R(r,t), the ratio of the areas B(r,t) and A(r,t) above and below the power function $f(x) = x^r$ $(0 \le x \le t)$, respectively, is simply r, the exponent of the power function f(x) that defines the boundary between B(r,t) and A(r,t).

Another observation can be made related to the power function $f(x) = x^r$ $(0 \le x \le t)$ which generates the rectangle R(r,t). The diagonal of R(r,t) with endpoints (0,0) and $\left(t, t^r\right)$ is defined by $d(x) = t^{r-1}x$ $(0 \le x \le t)$.

If $0 < r < 1$ then $-1 < r-1 < 0$. Since $0 \le x \le t$, then $0 \le t^{r-1} \le x^{r-1}$. Therefore $0 \le t^{r-1}x \le x^{r-1}x$, so that $0 \le d(x) \le f(x)$. Hence A(r,t) contains the diagonal d(x) of R(r,t), and so B(r,t) lies above d(x), except at the endpoints (0,0) and $\left(t, t^r\right)$. Thus A(r,t) > B(r,t) whenever $0 < r < 1$.

In a similar manner, if $r > 1$, then $r-1 > 0$. Therefore, since $0 \le x \le t$, then $0 \le x^{r-1} \le t^{r-1}$, so that $0 \le x^{r-1}x \le t^{r-1}x$, and so $0 \le f(x) \le d(x)$. Thus A(r,t) lies below the diagonal d(x) of R(r,t), except at the endpoints (0,0) and $\left(t, t^r\right)$, and so A(r,t) < B(r,t) whenever $r > 1$.

Finally, if $r = 1$ then the upper right vertex of the rectangle R(r,t) is (t,t). Furthermore, $f(x) = x^r = x = t^0x = t^{r-1}x = d(x)$. Thus $A(r,t) = \frac{1}{2} \cdot C(r,t) =$ B(r,t) whenever $r = 1$.

The facts of these three cases are illustrated in Figures 4, 5, and 6 below.

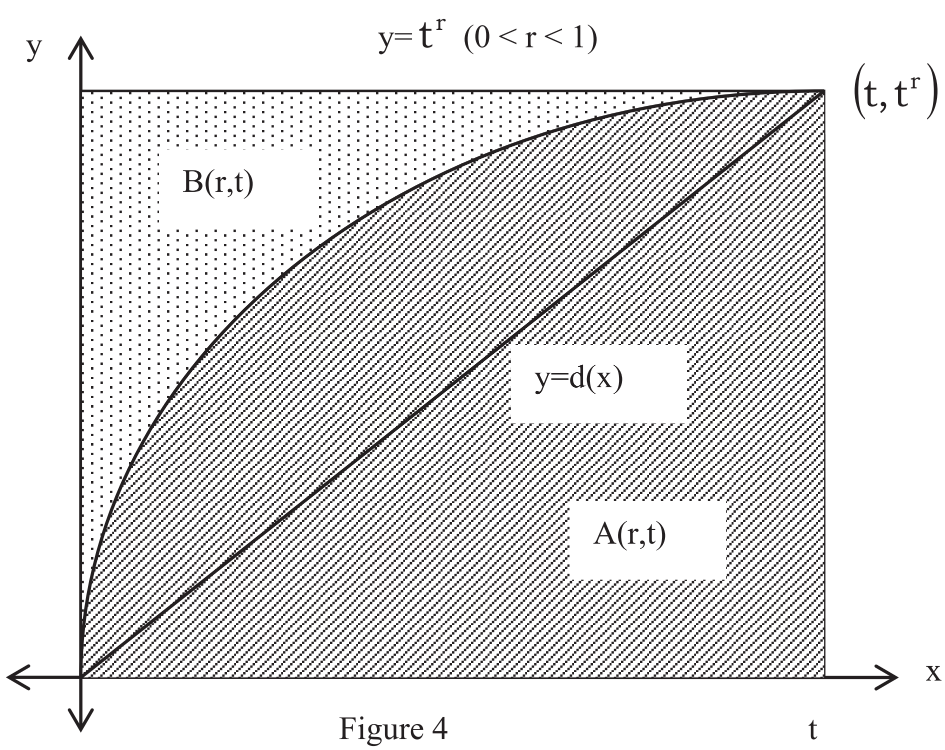

Figure 4

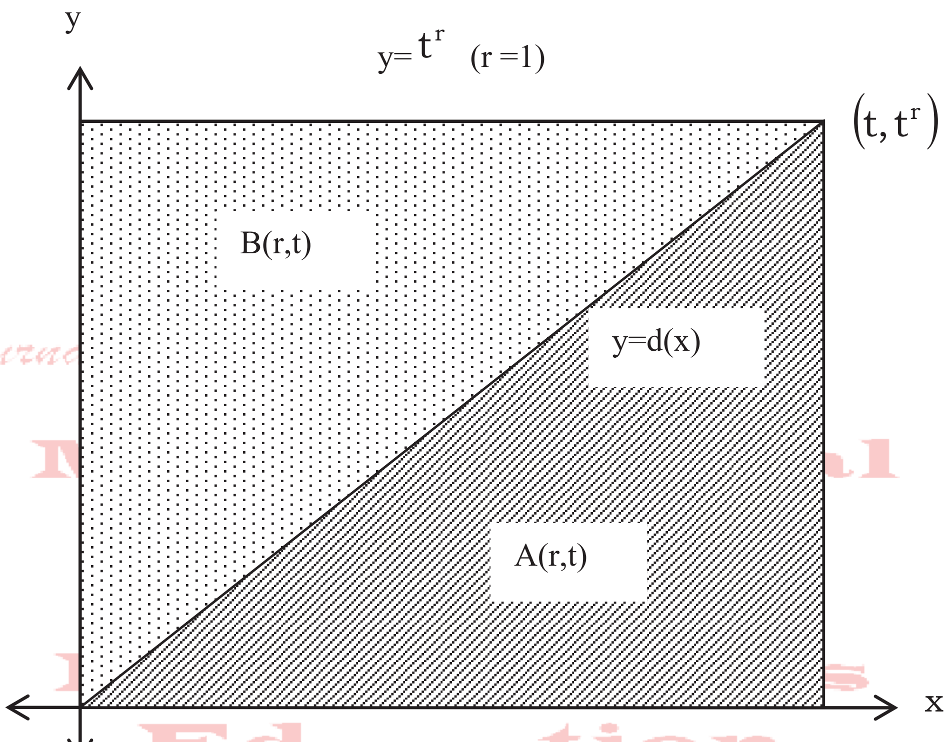

Figure 5

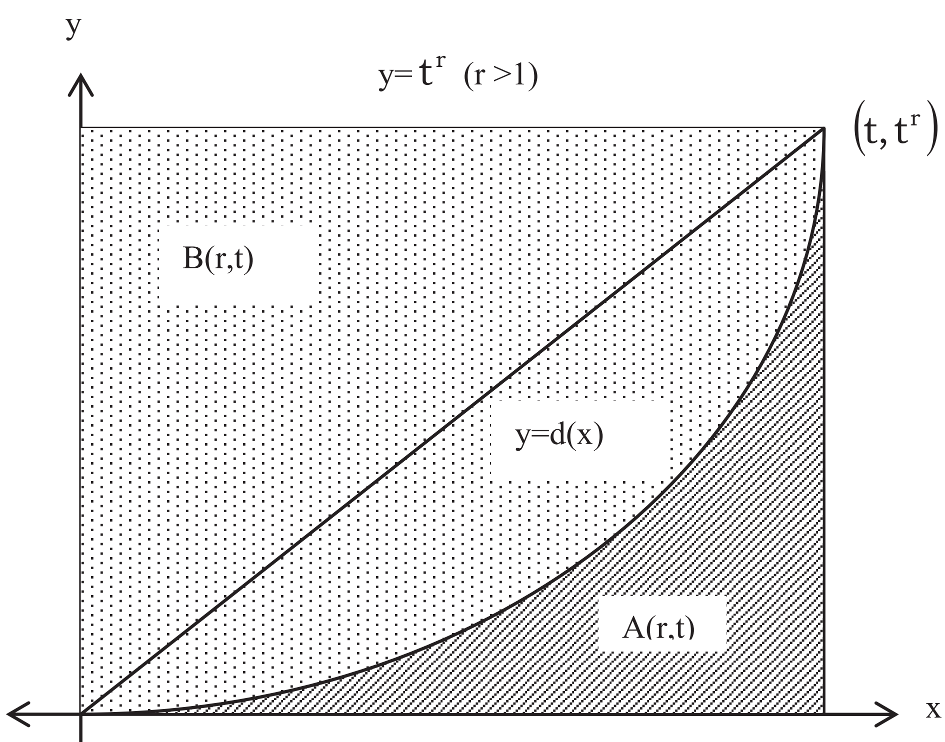

Figure 6

The facts of the three cases illustrated above in Figures 4, 5, and 6 can also be validated with the observation that since $\dfrac{B(r,t)}{A(r,t)} = r$ by (7), then B(r,t) = r·A(r,t). Clearly then

$$B(r,t)\begin{cases} < A(r,t) & \text{if } 0<r<1 \\ = A(r,t) & \text{if } r=1 \\ > A(r,t) & \text{if } r>1 \end{cases} \tag{8}$$

Limiting Case for r→0

For any value of t > 0, as r approaches 0, the graph of $f(x) = x^r$ $(0 \le x \le t)$ becomes asymptotic to the left and top boundaries of the rectangle R(r,t), which are defined by x = 0 and $y = t^r$, respectively, as shown in Figure 7 below.

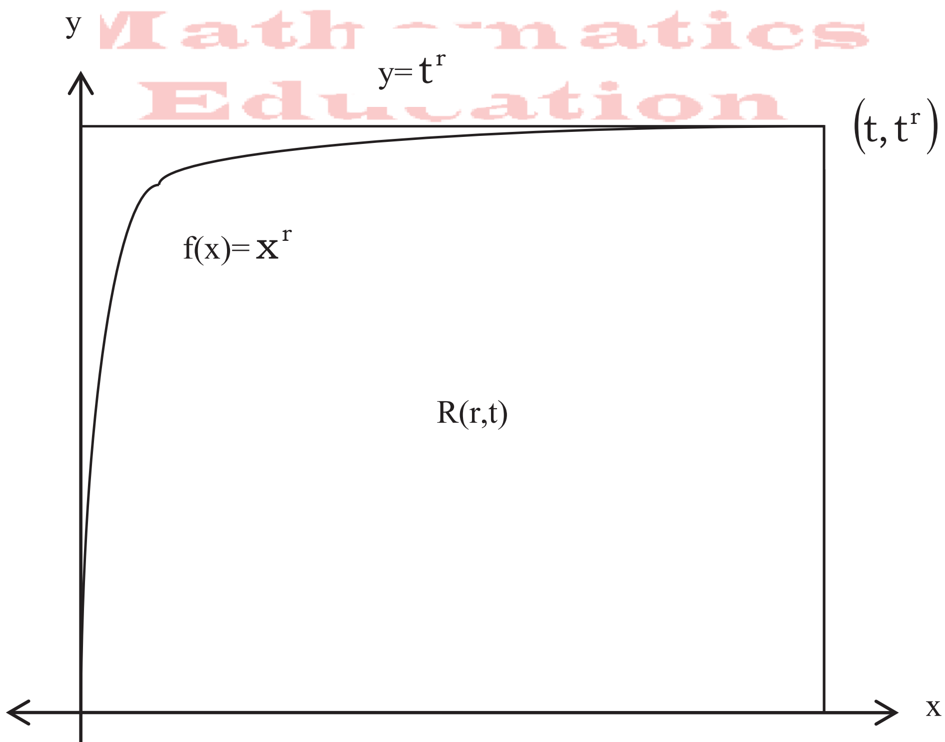

Figure 7

This graphical observation suggests that as r approaches 0, the area A(r,t) approaches C(r,t) and B(r,t) approaches 0. The first conjecture is verified by the fact that $\lim_{r\to 0} A(r,t) = \lim_{r\to 0} \dfrac{1}{r+1} t^{r+1}$ (by (1) above) $= t = \lim_{r\to 0} t^{r+1} = \lim_{r\to 0} C(r,t)$ (by (3) above). Furthermore, $\lim_{r\to 0} B(r,t) = \lim_{r\to 0} \dfrac{r}{r+1} t^{r+1}$ (by (2) above) $= 0 \cdot t = 0$.

Limiting Case for r→∞

On the other hand, for any value of $t > 0$, as r approaches ∞, the graph of $f(x) = x^r$ $(0 \le x \le t)$ becomes asymptotic to the right and bottom boundaries of the rectangle R(r,t), which are defined by $x = t$ and $y = 0$, respectively, as shown in Figure 8 below.

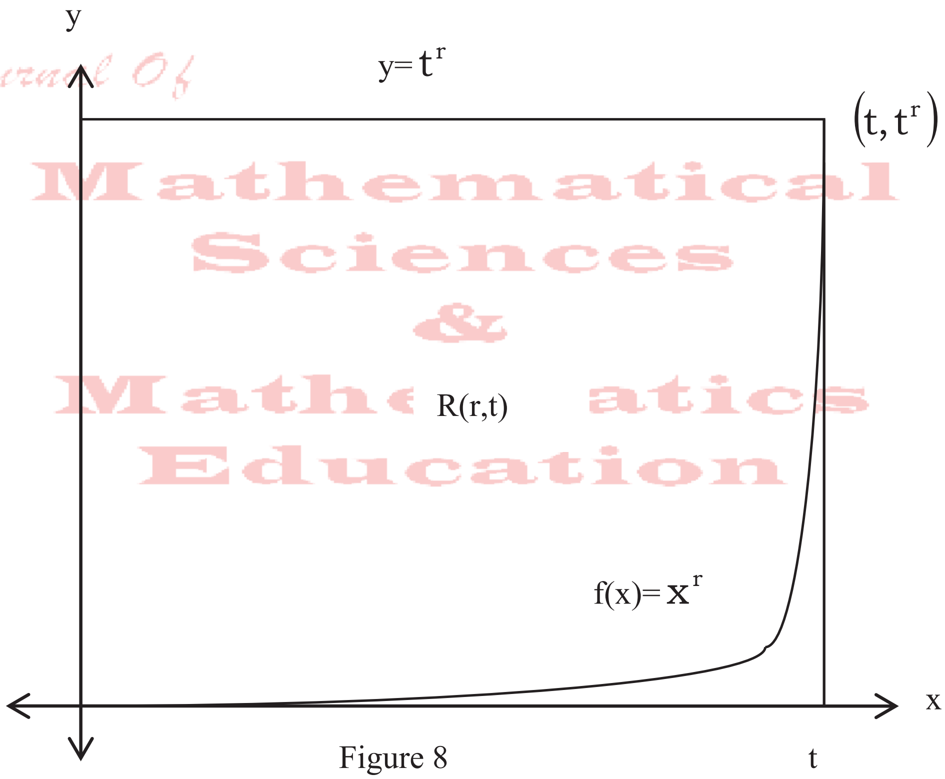

Figure 8

This graphical observation suggests that as r approaches ∞, the area B(r,t) approaches C(r,t) and A(r,t) approaches 0. However, this case is complicated by the fact that the upper right vertex of the rectangle R(r,t) has second coordinate t^r, which may be approaching 0, 1, or ∞ as r approaches ∞ depending on the value of t. Thus we are compelled to more closely examine A(r,t), B(r,t), C(r,t), and their relative areas in the following cases.

Case 1: Suppose that $0 < t < 1$.

Then $\lim_{r\to\infty} t^r = 0$ since $0 < t < 1$, and so $\lim_{r\to\infty}(t, t^r) = (t,0)$, where (t, t^r) is the upper right vertex of the rectangle R(r,t). Thus as r approaches ∞, the rectangle R(r,t) flattens against the x-axis with area C(r,t) approaching 0. Since $0 \le A(r,t) \le C(r,t)$ and $0 \le B(r,t) \le C(r,t)$, then A(r,t) and B(r,t) also approach 0 as r approaches ∞. These graphical observations are formally validated as follows.

Since $\lim_{r\to\infty} \frac{1}{r+1} = 0$ and $\lim_{r\to\infty} t^{r+1} = 0$ (since $0 < t < 1$), then $\lim_{r\to\infty} A(r,t) =$ $\lim_{r\to\infty} \frac{1}{r+1} t^{r+1}$ (by (1) above) $= \lim_{r\to\infty} \frac{1}{r+1} \cdot \lim_{r\to\infty} t^{r+1} = 0 \cdot 0 = 0$. Furthermore, $\lim_{r\to\infty} \frac{r}{r+1} = 1$, so that $\lim_{r\to\infty} B(r,t) = \lim_{r\to\infty} \frac{r}{r+1} t^{r+1}$ (by (2) above) $=$ $\lim_{r\to\infty} \frac{r}{r+1} \cdot \lim_{r\to\infty} t^{r+1} = 1 \cdot 0 = 0$. Finally, $\lim_{r\to\infty} C(r,t) = \lim_{r\to\infty} t^{r+1}$ (by (3) above) $= 0$.

Thus the graphical suggestion above that B(r,t) approaches C(r,t) and A(r,t) approaches 0 as r approaches ∞ is actually correct since all three areas approach 0.

Case 2: Suppose that $t = 1$.

Then $\lim_{r\to\infty} t^r = \lim_{r\to\infty} 1^r = 1$, and so the upper right vertex (t, t^r) of R(r,t) approaches $\lim_{r\to\infty} (t, t^r) = (1,1)$ as r approaches ∞. Consequently, the rectangle R(r,t) approaches the unit square with opposite vertices (0,0) and (1,1) and area $C(r,t) = 1$. Similar to the approach in Case 1 above, we pursue the following limits.

Since $\lim_{r\to\infty} \frac{1}{r+1} = 0$ and $\lim_{r\to\infty} t^{r+1} = \lim_{r\to\infty} 1^{r+1} = 1$, then $\lim_{r\to\infty} A(r,t) =$ $\lim_{r\to\infty} \frac{1}{r+1} t^{r+1}$ (by (1) above) $= \lim_{r\to\infty} \frac{1}{r+1} \cdot \lim_{r\to\infty} t^{r+1} = 0 \cdot 1 = 0$. Furthermore, $\lim_{r\to\infty} \frac{r}{r+1} = 1$, so that $\lim_{r\to\infty} B(r,t) = \lim_{r\to\infty} \frac{r}{r+1} t^{r+1}$ (by (2) above) $=$ $\lim_{r\to\infty} \frac{r}{r+1} \cdot \lim_{r\to\infty} t^{r+1} = 1 \cdot 1 = 1$. Finally, $\lim_{r\to\infty} C(r,t) = \lim_{r\to\infty} t^{r+1}$ (by (3) above) $=$ $\lim_{r\to\infty} 1^{r+1} = 1$.

Thus $\lim_{r\to\infty} B(r,t) = 1 = \lim_{r\to\infty} C(r,t)$ and $\lim_{r\to\infty} A(r,t) = 0$. Hence the graphical suggestion above that B(r,t) approaches C(r,t) and A(r,t) approaches 0 as r approaches ∞ is again correct.

Case 3: Suppose that $t > 1$.

Then $\lim_{r\to\infty} t^r = \infty$ since $t > 1$, and so the upper right vertex (t, t^r) of the rectangle R(r,t) becomes arbitrarily far above the x-axis. Thus as r approaches ∞, R(r,t) approaches infinite height with constant width t, so that the area C(r,t) of R(r,t) approaches ∞. Since $0 \le A(r,t) \le C(r,t)$ and $0 \le B(r,t) \le C(r,t)$, then A(r,t), B(r,t), or both may approach ∞ as r approaches ∞. In order to resolve these issues rigorously, we pursue the following limits similar to the approach in Case 1 and Case 2 above.

Since $\lim_{r\to\infty}(r+1) = \infty$ and $\lim_{r\to\infty} t^{r+1} = \infty$ (since t > 1), then $\lim_{r\to\infty} A(r,t) =$

$\lim_{r\to\infty} \frac{1}{r+1} t^{r+1}$ (by (1) above) $= \lim_{r\to\infty} \frac{t^{r+1}}{r+1} = \lim_{r\to\infty} \frac{\frac{d}{dr} t^{r+1}}{\frac{d}{dr}(r+1)}$ (by L'Hospital's Rule)

$= \lim_{r\to\infty} t^{r+1} \cdot \ln t = \ln t \cdot \lim_{r\to\infty} t^{r+1} = \infty$ (since t > 1 implies that $\lim_{r\to\infty} t^{r+1} = \infty$ and

$\ln t > 0$). Furthermore, $\lim_{r\to\infty} B(r,t) = \lim_{r\to\infty} \frac{r}{r+1} t^{r+1}$ (by (2) above) $= \lim_{r\to\infty} \frac{r \cdot t^{r+1}}{r+1} =$

$\lim_{r\to\infty} \frac{\frac{d}{dr} r \cdot t^{r+1}}{\frac{d}{dr}(r+1)}$ (by L'Hospital's Rule) $= \lim_{r\to\infty} t^{r+1}(1 + r \cdot \ln t) = \infty$ (since t > 1

implies that $\lim_{r\to\infty} t^{r+1} = \infty$ and $\lim_{r\to\infty} (1 + r \cdot \ln t) = \infty$). Finally, $\lim_{r\to\infty} C(r,t) = \lim_{r\to\infty} t^{r+1}$

(by (3) above) = ∞ since t > 1.

Therefore the conjecture motivated by the graphical evidence above that B(r,t) approaches C(r,t) and A(r,t) approaches 0 as r approaches ∞ cannot be validated in this case since all three limits diverge. However, we can achieve a sense of these relationships through the relative areas established in (5), (6), and (7). With this approach, we have $\lim_{r\to\infty} \frac{B(r,t)}{C(r,t)} = \lim_{r\to\infty} \frac{r}{r+1}$ (by (6) above) = 1,

$\lim_{r\to\infty} \frac{A(r,t)}{B(r,t)} = \lim_{r\to\infty} \frac{1}{r}$ (by (7) above) = 0, and $\lim_{r\to\infty} \frac{A(r,t)}{C(r,t)} = \lim_{r\to\infty} \frac{1}{r+1}$ (by (5) above) = 0. Thus the area B(r,t) relative to C(r,t) approaches 1 as r approaches ∞. Furthermore, the area A(r,t) relative to either B(r,t) or C(r,t) approaches 0 as r approaches ∞. Hence some degree of validation for the conjecture stated above that B(r,t) approaches C(r,t) and A(r,t) approaches 0 as r approaches ∞ is established for the case in which t > 1.

Concluding Remarks

As mentioned above, the 2005 result by Winton on the centroid of the finite region bounded by a power function and its inverse [1, pp. 228–231] constitutes a good undergraduate research project for a calculus student. Furthermore, the specific application to power functions can easily be extended to other types of functions and their inverses, provided that the regions bounded are finite and reasonable.

Similarly, the results of this paper are specific to power functions. Thus an analogous investigation for a different class of functions or a specific function from a different class could be used as an undergraduate research project for a calculus student. Consequently, the results of this paper and the 2005 paper by

Winton which are specific to power functions can possibly be generalized to include other functions.

The 2010 result by Winton and Warren for the generalized surface area of revolution of an arbitrary continuous function about an arbitrary linear axis in the Cartesian plane [2, pp. 25–33] can also be used as an undergraduate research project for a calculus student. However, in contrast to the two preceding projects, the more general result of that paper can be investigated for special cases of specific functions or particular linear axes of revolution.

† *Richard Winton*, *Ph.D.*, Tarleton State University, USA
‡ *Sarah S. Horner*, *PE*, Texas Department of Transportation, USA

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1. Winton, Richard A., *Symmetry at Infinity*, College Mathematics Journal, Vol. 36, No. 3, (2005) pp. 228–231.
2. Winton, Richard A. and Warren, Michael L., *Generlized Surface Area of Revolution*, Journal of Mathematical Sciences and Mathematics Education, Vol. 5, No. 1, (2010) pp. 25–33.