Polygons on sides of octagons

Rogério César dos Santos, Ph.D. †

Abstract

Van Aubel's theorem has some interesting generalizations. Some were dealt with by Krishna (2018a and 2018b). In this article, we intend, encouraged by the works cited, to prove a new generalization of Van Aubel's theorem, which consists of the construction of a parallelogram from an octagon surrounded by regular n-sided polygons.

The present article will demonstrate the following result about octagons, driven by the results of Krishna (2018*a*) and Krishna (2018*b*):

Theorem: Take a random octagon $O: A_1A_2 \dots A_8$, either convex or concave. Having fixed an integer $n \geq 3$, consider the eight *regular* polygons of n sides constructed externally on the sides of O. Denote the sixteen sides of the eight polygons that have the common vertex A_j with the octagon, $j = 1, \dots, 8$, by $A_1B_{82}, A_1B_{11}, A_2B_{12}, A_2B_{21}, \dots, A_8B_{72}$ and A_8B_{81}, as illustrated in figure 1. Denote the midpoints of these sides by $C_{11}, C_{12}, C_{21}, C_{22}, \dots, C_{81}$ and C_{82}, respectively. The figure shows only three of the regular polygons so that the image is not overloaded.

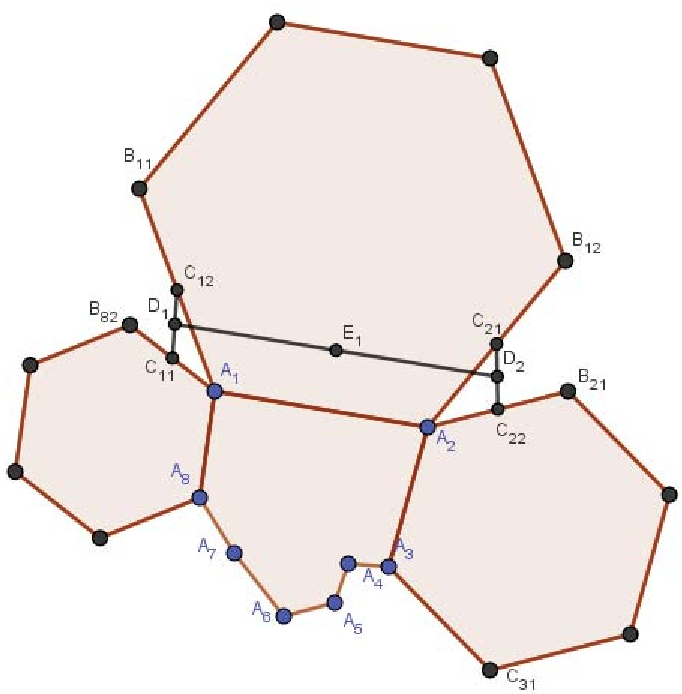

Figure 1 - Regular polygons of n sides on sides of a random octagon

Also consider

$$D_j = \frac{C_{j1} + C_{j2}}{2}, j = 1, \dots, 8 \mod 8$$
the midpoints of segments $C_{j1}C_{j2}$ and
$$E_j = \frac{D_j + D_{j+1}}{2}$$
the midpoints of $D_j D_{j+1}$.

Under these conditions, the midpoints of $E_j E_{j+4}$, $j = 1, \dots, 8$ mod 8, denoted by I, J, K and L, respectively, form a parallelogram, as shown in figure 2.

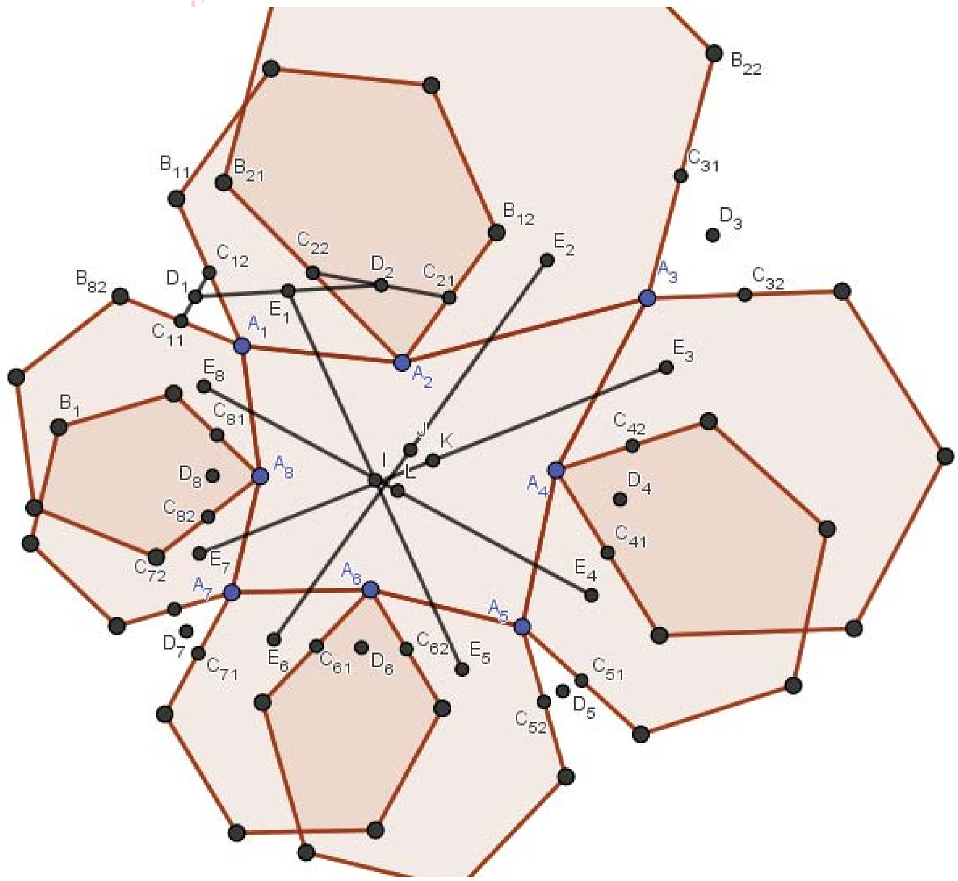

Figure 2 - The parallelogram $IJKL$

Proof: Let us demonstrate it by denoting the points I, J, K and L as a function of the coordinates of the vertices of the octagon. Considering $A_j = (x_j, y_j)$ each vertex of the octagon in the Cartesian Plane, $j = 1, \dots, 8$, denote the complex number corresponding to each A_j by $a_j = x_j + iy_j$, where $i^2 = -1$. Similarly, denote the complex numbers corresponding to the points B_j, C_j, D_j and E_j by b_j, c_j, d_j and e_j, respectively. Take M_j as the midpoint of the segment $A_j A_{j+1}$, so that $m_j = \frac{a_j + a_{j+1}}{2}$ is the complex number corresponding to M_j.

Considering $\hat{A} = B_{11}\hat{A}_1A_2$ the internal angle of each of the eight regular polygons, then, through the hypotheses, we can conclude that $c_{12} - a_1$ is equal to the complex number resulting from the rotation of $m_1 - a_1$ by $\hat{A}$ degrees counterclockwise. Namely,

$$c_{12} - a_1 = (m_1 - a_1)(\cos\hat{A} + i\sin\hat{A}).$$

From now on let us denote $\text{cis}(\hat{A}) = \cos\hat{A} + i\sin\hat{A}$. Then,

$$c_{12} = a_1 + (m_1 - a_1)\text{cis}(\hat{A}).$$

The complex number $c_{11} - a_1$ is equal to the complex number resulting from the rotation of $m_8 - a_1$ by $\hat{A}$ degrees clockwise, or $360^0 - \hat{A}$ counterclockwise, that is,

$$c_{11} - a_1 = (m_8 - a_1)\text{cis}(360^0 - \hat{A}) \Rightarrow$$

$$c_{11} - a_1 = (m_8 - a_1)[\cos(360^0 - \hat{A}) + i\sin(360^0 - \hat{A})] =$$

$$(m_8 - a_1)[\cos\hat{A} - i\sin(\hat{A})] =$$

$$(m_8 - a_1)[\cos(-\hat{A}) + i\sin(-\hat{A})] =$$

$$(m_8 - a_1)\text{cis}(-\hat{A})$$

$$\Rightarrow$$

$$c_{11} = a_1 + (m_8 - a_1)\text{cis}(-\hat{A}).$$

Therefore, we can find

$$d_1 = \frac{c_{11} + c_{12}}{2} = \frac{1}{2}[a_1 + (m_8 - a_1)\text{cis}(-\hat{A}) + a_1 + (m_1 - a_1)\text{cis}(\hat{A})] =$$

$$\frac{1}{2}\left[a_1 + \left(\frac{a_8 + a_1}{2} - a_1\right)\text{cis}(-\hat{A}) + a_1 + \left(\frac{a_1 + a_2}{2} - a_1\right)\text{cis}(\hat{A})\right] =$$

$$\frac{1}{2}\left[2a_1 + \frac{a_8 - a_1}{2}\text{cis}(-\hat{A}) + \frac{a_2 - a_1}{2}\text{cis}(\hat{A})\right].$$

Similarly, we can find

$$d_2 = \frac{c_{21} + c_{22}}{2} =$$

$$\frac{1}{2}\left[2a_2 + \frac{a_1 - a_2}{2}\text{cis}(-\hat{A}) + \frac{a_3 - a_2}{2}\text{cis}(\hat{A})\right].$$

Then,

$$e_1 = \frac{d_1 + d_2}{2} = \frac{1}{4} \times$$

$$\left[2a_1 + \frac{a_8 - a_1}{2}\text{cis}(-\hat{A}) + \frac{a_2 - a_1}{2}\text{cis}(\hat{A}) + 2a_2 + \frac{a_1 - a_2}{2}\text{cis}(-\hat{A}) + \frac{a_3 - a_2}{2}\text{cis}(\hat{A})\right]$$

$$= \frac{1}{4} \times \left[2(a_1 + a_2) + \frac{a_8 - a_2}{2}\text{cis}(-\hat{A}) + \frac{a_3 - a_1}{2}\text{cis}(\hat{A})\right].$$

Similarly, we obtain

$$e_5 = \frac{1}{4} \times \left[2(a_5 + a_6) + \frac{a_4 - a_6}{2}\text{cis}(-\hat{A}) + \frac{a_7 - a_5}{2}\text{cis}(\hat{A})\right].$$

Thus,

$$I = \frac{e_1 + e_5}{2} = \frac{1}{8} \times$$

$$\left[2(a_1 + a_2 + a_5 + a_6) + \frac{a_8 + a_4 - a_2 - a_6}{2}\operatorname{cis}(-\hat{A}) + \frac{a_3 + a_7 - a_1 - a_5}{2}\operatorname{cis}(\hat{A})\right].$$

Similarly, we can obtain:

$$J = \frac{e_2 + e_6}{2} = \frac{1}{8} \times \left[2(a_2 + a_3 + a_6 + a_7) + \frac{a_1 + a_5 - a_3 - a_7}{2}\operatorname{cis}(-\hat{A}) + \frac{a_4 + a_8 - a_2 - a_6}{2}\operatorname{cis}(\hat{A})\right].$$

$$K = \frac{e_3 + e_7}{2} = \frac{1}{8} \times \left[2(a_3 + a_4 + a_7 + a_8) + \frac{a_2 + a_6 - a_4 - a_8}{2}\operatorname{cis}(-\hat{A}) + \frac{a_5 + a_1 - a_3 - a_7}{2}\operatorname{cis}(\hat{A})\right]$$

and

$$L = \frac{e_4 + e_8}{2} = \frac{1}{8} \times \left[2(a_4 + a_5 + a_8 + a_1) + \frac{a_3 + a_7 - a_5 - a_1}{2}\operatorname{cis}(-\hat{A}) + \frac{a_6 + a_2 - a_4 - a_8}{2}\operatorname{cis}(\hat{A})\right].$$

Anyway, let us calculate $|J - K| = \frac{1}{8} \times$

$$\left|2(a_2 + a_6 - a_4 - a_8) + \frac{a_1 - a_2 - a_3 + a_4 + a_5 - a_6 - a_7 + a_8}{2}\operatorname{cis}(-\hat{A}) + \frac{-a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_7 + a_8}{2}\operatorname{cis}(\hat{A})\right|.$$

Remembering that

$$I = \frac{1}{8} \times \left[2(a_1 + a_2 + a_5 + a_6) + \frac{a_8 + a_4 - a_2 - a_6}{2}\operatorname{cis}(-\hat{A}) + \frac{a_3 + a_7 - a_1 - a_5}{2}\operatorname{cis}(\hat{A})\right]$$

and

$$L = \frac{1}{8} \times \left[2(a_4 + a_5 + a_8 + a_1) + \frac{a_3 + a_7 - a_5 - a_1}{2}\operatorname{cis}(-\hat{A}) + \frac{a_6 + a_2 - a_4 - a_8}{2}\operatorname{cis}(\hat{A})\right],$$

we have:

$$|I - L| = \frac{1}{8} \times$$

$$\left|2(a_2 + a_6 - a_4 - a_8) + \frac{a_1 - a_2 - a_3 + a_4 + a_5 - a_6 - a_7 + a_8}{2}\operatorname{cis}(-\hat{A}) + \frac{-a_1 - a_2 + a_3 + a_4 - a_5 - a_6 + a_7 + a_8}{2}\operatorname{cis}(\hat{A})\right| = |J - K|.$$

In an analogous way, it is possible to prove that $|K - L| = |J - I|$, which concludes the demonstration.

† *Rogério César dos Santos, Ph.D.*, University of Brasília - Brazil

References

Krishna, Dasari Naga Vijay. A New Equilateral Triangle associated with Hexagon. 2018*a*.

Krishna, Dasari Naga Vijay . A note on special cases of Van Aubel's theorem. International Jounal of Advances in Applied Mathematics and Mechanics 5(4). 2018*b*.