On Frechet Derivatives with Application to the Inverse Function Theorem of Ordinary Differential Equations

Eziokwu, C. Emmanuel, Ph.D. †

Abstract

In this paper, the Frechet differentiation of functions on Banach space was reviewed. We also investigated it's algebraic properties and its relation by applying the concept to the inverse function theorem of the ordinary differential equations. To achieve the feat, some important results were considered which finally concluded that the Frechet derivative can extensively be useful in the study of ordinary differential equations.
Key Words: Banach space, Frechet differentiability Lipchitz function Inverse function theorem.

Introduction

The Frechet derivative, a result in mathematical analysis is derivative usually defined on Banach spaces. It is often used to formalize the functional derivatives commonly used in physics, particularly quantum field theory. The purpose of this work is to review some results obtained on the theory of the derivatives and apply it to the inverse function theorem in ordinary differential equations.

1.0 Derivatives

Definition 1.1 [Kaplon (1958)]:
Let f be an operator mapping a Banach space X into a Banach space Y. If there exists a bounded linear operator T from X into Y such that:

$$\lim_{\|\Delta x\|\to 0} \frac{\|f(x_0+\Delta x)-f(x_0)-T(\Delta x)\|}{\|\Delta x\|} = 0 \qquad (1.1)$$

Or,

$$\lim_{t\to 0} \frac{\|f(x+th)-f(x)\|}{\|t\|} = T_x(h)$$

Or

$$\lim_{y\to 0} \frac{\|f(x+y)-f(x)-\mathrm{T}(y)\|}{\|y\|} = 0 \qquad (1.1)$$

then P is said to be Frechet differentiable at x_0, and the bounded linear operator

$$P'(x_0) = T \qquad (1.2)$$

is called the first Frechet – derivative of f at x_0. The limit in (1.1) is supposed to hold independent of the way that Δx approaches 0. Moreover, the Frechet differential

$$\delta f\ (x_0, \Delta x) = f'(x_0)\ \Delta x$$

is an arbitrary close approximation to the difference $f\ (x_0 + \Delta x) - f\ (x_0)$ relative to $\|\Delta x\|$, for $\|\Delta x\|$ small.

If f_1 and f_2 are differentiable at x_0, then

$$(f_1 + f_2)'(x_0) = f_1'(x_0) + f_2'(x_0)$$

Moreover, if f_2 is an operator from a Banach space X into a Banach space X into a Banach space Z, and f_1 is an operator from Z into a Banach space Y, their composition $f_1 \circ f_2$ is defined by

$$(f_1 \circ f_2)\ (x) = f_1(f_2(x)),\ \ for\ all\ x \in X$$

We know that $f_1 \circ f_2$ is differentiable at x_0 if f_2 is differentiable at x_0 and f_1 is differentiable at f_2 (x_0) of Z, with (chain rule):

$$(f_1 \circ f_2)'(x) = f_1'\left(f_2(x_0)\right) f_2'(x_0)$$

In order to differentiate an operator f we write:

$$f(x_0 + \Delta x) - f(x_0) = T(x_0, \Delta x)\Delta x + \eta(x_0, \Delta x),$$

where $T\ (x_0, \Delta x)$ is a bounded linear operator for given $x_0, \Delta x$ with

$$\lim_{\|\Delta x\| \to 0} T(x_0, \Delta x) = T \qquad (1.3)$$

and

$$\lim_{\|\Delta x\| \to 0} \frac{\|\eta\ (x_0, \Delta x)\|}{\|\Delta x\|} = 0 \qquad (1.4)$$

Estimate (1.3) and (1.4) give

$$\lim_{\|\Delta x\| \to 0} T(x_0, \Delta x) = f'$$

If $T(x_0, \Delta x)$ is a continuous function of Δx in some ball $\cup\ (0, R), (R > 0)$, then

$$T\ (x_0, 0) = f'(x_0)$$

We now present the definition of a mosaic:

Higher – order derivatives can be defined by induction:

Definition 1.2 [Argyros (2005)]

If f is (m – 1) – times Frechet – differentiable ($m \geq 2$ an integer), and an m – linear operator A from X into Y exists such that

$$\lim_{\|\Delta x\| \to 0} \frac{\left\|f^{(m-1)}\ (x_0 + \Delta x) - f^{(m-1)}(x_0) - A(\Delta x)\right\|}{\|\Delta x\|} = 0 \qquad (1.5)$$

then A is called the *m* – Frechet – derivative of f at x_0, and

$$A = f^{(m)}\ (x_0) \qquad (1.6)$$

Definition 1.3 [Koplan (1958)]

Suppose $f: U \subseteq R^n \rightarrow R^m$ where U is an open set, the function f is classically differentiable at $x_0 \in U$ if
The partial derivatives of

$f, \frac{\partial f_i}{\partial x_j}$ $for\ i = 1 \dots m\ and\ j = 1 \dots n\ exists\ at\ x_0,$

The Jacobean matrix $J(x_0) = \left[\frac{\partial f_i}{\partial x_j}(x_0) \in R^{m\,x\,n}\right]$ satisfies

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - J(x_o)(x - x_0)\|}{\|x - x_0\|} = 0 .$$

We say that the Jacobean matrix $J(x_0)$ is the derivative of $f\ at\ x_0$, that is called total derivative

Higher partial derivatives in product spaces can be defined as follows:
Define

$$X_{ij} = T(X_j, X_i) \qquad (1.7)$$

where $X_1, X_2, \dots$ are Banach spaces and $T(X_j, X_i)$ is the space of bounded linear operators from X_j into X_i. The elements of X_{ij} are denoted by T_{ij}, etc.
Similarly,

$$X_{ijm} = T(X_m, X_{ij}) = T(X_{jk}, X_{ij1j2} \dots jm - 1), \qquad (1.8)$$

which denotes the space of bonded linear operators from X_{jm} into $X_{ij1j2} \dots jm - 1$. The elements $A = A_{ij1j2} \dots jm$ are a generalization of m – linear operators.
Consider an operator f_1 from space

$$X = \prod_{p=1}^{n} x_{jp} \qquad (1.9)$$

into X_i, and that f_i has partial derivative of orders $1, 2, \dots, m - 1$ in some ball $\cup(x_0, R)$, where $R > 0$ and

$$x_0 = \left(x_{j1}^{(0)}, x_{j2}^{(0)}, \dots, x_{jn}^{(0)}\right) \in X$$

For simplicity and without loss of generality we [Frechet (1906)] remember the original spaces so that

$$j_1 = 1, j_2 = 2, \dots, j_n = n$$

hence, we write

$$x_0 = \left(x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}\right)$$

A partial derivative of $order\ (m - 1)$ of f_i at x_0 is an operator

$$A_{iq1q2\dots qm-1} = \frac{\partial^{(m-1)} f_i(x_0)}{\partial x_{q1}\, \partial x_{q2} \dots \partial x_{qm-1}} \qquad (1.10)$$

$(in\ X_{iq1q2\ldots qm-1})$ where
$1 \leq q_1, q_2, \ldots, q_{m-1} \leq n$
Let $P(X_{qm})$ denote the operator from X_{qm} into $X_{iq1\,q2\ldots qm-1}$ obtained by letting
$x_j = x_j^{(0)}, \qquad j \neq q_m$
for some $q_m, 1 \leq q_m \leq n.$ Moreover, if

$$P'\left(x_{qm}^{(0)}\right) = \frac{\partial}{\partial x_{qm}} \cdot \frac{\partial^{m-1} f_i(x_0)}{\partial x_{q1} \partial x_{q2} \ldots \partial x_{qm-1}} = \frac{\partial^m f_i(x_0)}{\partial x_{q1} \ldots \partial x_{qm}}, \qquad (1.11)$$

exists it will be called the partial Frechet – derivative of order m of f_i with respect to $x_{q1}, \ldots, x_{qm}$ at x_0

Furthermore, if f_i is Frechet – differentiable m times at x_0, then

$$\frac{\partial^m f_i(x_0)}{\partial x_{q1} \ldots \partial x_{qm}} x_{q1} \ldots x_{qm} = \frac{\partial^m f_i(x_0)}{\partial x_{s1} \partial x_{q2} \ldots \partial x_{sm}} x_{s1} \ldots x_{sm} \qquad (1.12)$$

For any permutation $s_1, s_2, \ldots, s_m$ of integers $q_1, q_2, \ldots, q_m$ and any choice of point $x_{q1}, \ldots, x_{qm}$, from $X_{q1}, \ldots, X_{qm}$ respectively. Hence, if $f = (f_1, \ldots, f_t)$ is an operator from $X = X_1 + X_2 + \ldots + X_n$ into $Y = Y_1 + Y_2 + \ldots + Y_t$ then

$$f^{(m)}(x_0) = \left(\frac{\partial^m f_i}{\partial x_{j1} \ldots \partial x_{jm}}\right)_{x = x_0} \qquad (1.13)$$

$i = 1, 2, \ldots, t, j_1, j_2, \ldots, j_m = 1, 2, \ldots n$ is called the m – Frechet derivative of F at $x_0 = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$.

1.1 Integration

In this subsection we [Pantryagin (1962)] state results concerning the mean value theorem. Taylor's theorem, and Riemannian integration without the proofs. The mean value theorem for differentiable real functions f:

$f(b) - f(a) = f'(c)(b - a),$

Where c $\in (a, b)$, does not hold in a Banach space setting. However, if F is a differentiable operator between two Banach spaces X and Y, then

$$\|f(x) - f(y)\| \leq \sup_{\bar{x} \in L(x,y)} \|f'(\bar{x})\| . \|x - y\|$$

where

$$T(x,y) = \{z : z = \lambda y + (1-\lambda)x, 0 \leq \lambda \leq 1\}$$

Set

$$z(\lambda) = \gamma_y + (1-\lambda)x, 0 \leq \lambda \leq 1,$$

Divide the interval $0 \leq \lambda \leq 1$ into n subintervals of lengths $\Delta\lambda_i, i = 1, 2, \dots, n,$ choose points λ_i inside corresponding subintervals and as in the real Riemann integral consider sums

$$\sum_{\sigma} f(\lambda_i)\Delta\lambda_i = \sum_{i=1}^{n} f(\lambda_i)\Delta\lambda_i$$

where σ is the partition of the interval, and set

$$|\sigma| = \max_{(i)} \Delta\lambda_i$$

Definition 1.4 [ince (1956)]

If

$$S = \lim_{|\sigma|\to 0} \sum_{\sigma} f(\lambda_i)\Delta\lambda_i$$

exists, then it is called the Riemann integral from $f(\lambda)$ on [0,1], denoted by

$$S = \int_0^1 f(\lambda)d\lambda = \int_x^y f(\lambda)d\lambda$$

Definition 1.5 [Dunford and Schwartz (1958)]

A bounded operator $P\ (\lambda)$ on $[0,1]$ such that the set of points of discontinuity is of measure zero is said to be integrable on $[0,1]$.

Theorem 1.1 [Day (1973)]

If F is m – times Frechet – differentiable in $U(x_0, R), R > 0$, and $f^{(m)}\ (x)$ is integrable from x to any $\in \cup\ (x_0, R)$ then

$$f(y) = f(x) + \sum_{n=1}^{m-1} \frac{1}{n!} f^{(n)}(x)(y-x)^n + R_m(x,y), \qquad (1.14)$$

$$\left\| f(y) - \sum_{n=0}^{m-1} \frac{1}{n!} f^{(n)}(x)(y-x)^n \right\| \leq \sup_{\bar{x} \in L\ (x,y)} \left\| f^{(m)}(\bar{x}) \right\| \frac{\|y-x\|^m}{m!}, \qquad (1.15)$$

where

$$R_m(x,y) = \int_0^1 f^{(m)}(\lambda y + (1-\lambda)x)(y-x)^m \frac{(1-\lambda)^{m-1}}{(m-1)!} d\lambda \qquad (1.16)$$

2.0 Theorem and Properties

Definition 2.1 [Arthanasius (1973)]

Assume that X, Y are Banach spaces, S is an open subset of X and $f: S \to Y$, is continuous and Frechet differentiable at every point of S, Moreover assume that for every $\in > 0$ there exists $\delta(\in) > 0$ such that If $x_1, x_2, \in S$ with $\|x_1 - x_2\| \leq (C \in)$ then

$\|[f'(x_1) - f(x_2)]h\| \leq \in h \ \forall \ \|h\| \in X$

$\|\propto (x, L)\| \leq \propto [h]$ for every $x \in S, h \in X$ with $h \leq b(\in)$ then f is called $C-$differentiable on S

Theorem 2.1 [Arthanasius (1973)]

Let f be a convex function defined on an open convex subset X of a Banach space X that is continuous at $x \in X.$ then f is frechet differentiable at x iff

$$\lim_{t \to 0} \frac{\|f(x+th) + f(x-th) - 2f(r)\|}{\|t\|} = D$$

$uniformly\ for\ h \in S_x$

Remark 2.1

Obviously, we have earlier seen that Frechet differentiability has additive property and the product of two Frechet differentiable functions is Frechet differentiable function (the function that is Frechet differentiable is continuous and therefore locally bounded). Now we use boundedness of Frechet derivative and triangle inequality. A function which is Frechet differentiable at a point is continuous their.

3 THE INVERSE FUNCTION THEOREM

The inverse function theorem is an important tool in the theory of differential equations It ensures the existences of solution of the equation $Tx = y$. Although T is not assumed to be compact and the contraction principle might not be directly applicable, it is shown, in the proof of the inverse function theorem, that the contraction principle can be used indirectly if T has some appropriate differentiability properties.

Definition 3.1 [Arthanasius (1973)]

Let S be an open subset of the Banach space X and let f $map\ S$ into the Banach space Y. Fix a point $x_0 \in S$ $and\ let\ f(x_0) = x_0,\ then,$ f is said to be "locally invertible at (x_0, y_0) if there exist two numbers

$\propto > 0,\ \beta > 0,$ with the following property: for every y $\in\ S,\ S_0\ (y_0)$ there exists a unique $x\ \in\ S_0(x_0)$ such that $f\ (x) = y.$

Lemma 3.1 (uniqueness property of Frechet derivative) [Kaplan (1958)]

Let $L\ S\ \to Y$ be given with S an open subset of the Banach space $X\ and\ Y$ another Banach space. Suppose further that f is Frechet differentiable at $x \in S$. Then the Frechet derivative of $f\ at\ x$ is unique.

Proof. Suppose that $D_1\ (x), D_2\ (x)$are Frechet derivatives of $f\ at\ x$ with remainders $\omega_1(x_1,h),\ \ \omega_2\ (x_2\ h)$ respectively. Then we have

$$D_1\ (x)h + \ \omega_1\ (x,h) = \ D_2(x)h + \ \omega_2(x,h)$$

for every $h\ \in X\ with\ x + h\ \in\ S_1.\ Here\ S_1$ are some open subsets of S containing x. It follows that

$$\|D_1\ (x)h - \ D_2\ (x)h\ \|/\|h\| = \ \|\omega_1(x,h) - w_2(x,h)\| \qquad (3.1)$$

$$1/\leq\ \|\omega_1(x,h)\ \|h\| + \ \omega_2(x,h)\|/\|h\|$$

The last member of (3.1) tends to zero as $\|h\| \to\ 0.$

Let

$$T_x = [D_1\ (x) - D_2(x)]x, \qquad x\ \in X$$

then T is a linear operator on X such that

$$\lim_{\|x\| - 0} \|Tx\|\ /\|x\|\ \ = 0$$

Consequently given $\in\ > 0,$ there exists $\delta\ (\in) > 0$ such that $\|Tx\|/\|x\| >\ \epsilon$ for every $x\ \in X\ with\ \|x\| <\ \delta(\in).Given\ y\ \in X\ with\ y\ \neq 0, let\ x = \ \delta(\in)y\ /\ 2\ \|y\|$

. Then $\|\infty\| <\ \delta(\in)\ and \frac{\|Tx\|}{\|x\|}\ <\ \epsilon\ or\ \|Ty\| <\ \epsilon\|y\|.Since\ \epsilon$ is arbitrary, we obtain $Ty = 0\ for\ every\ y\ \in X.$ we conclude that

$$D_1\ (x) = \ D_2\ (x).$$

The existence of a bounded Frechet derivative $f\ u$ is equivalent to the continuity of $f\ at\ x.$ this is the content of the next lemma.

Lemma 3.2 [Argyros (2005)]

Let $f\colon S\ \to Y$ be given where S is an open subset of a Banach space $X\ and\ Y$ another Banach space Let f be Frechet differentiable at $x\ \ \in S$. Then f is continuous at x if $f'\ (x)$ is a bounded linear operator .

Proof. Let f be continuous at $x\ \in\ S$. Then for each $\epsilon\ > 0$ there exists $\delta\ (\epsilon) \in\ (0,1)$such that

$$\|f(x + h\ ) - \ f\ (x)\| <\ \in/2\,, \|f(x + h\ ) - \ f\ (x)h\| < (\epsilon/2)\|h\| < \epsilon/2$$

for all $h\ \in B\ with\ x + h\ \in\ S\ and\ \|h\| < \delta(\in).$

Therefore.

$\|f'(x)h\| < (\epsilon/2)\|h\| + \| f(x+h) - f(x)\| < \epsilon$

For $\|h\| < \delta(\epsilon), x+h \in S,$ it follows that the linear operator $f'(x)$ is continuous at the point 0.

We should note here that the magnitude of the ball S plays no role in the Frechet differentiability of f. This means that, to define the frechet derivative $f'(x) of\ at\ x,$ we only need to have that certain differentiability conditions hold for all $x+h$ in a sufficiently small open neighborhood $S\ of\ the\ point\ x$.

We now quote a well known theorem of functional analysis the "bounded inverse theorem"

Theorem 3.2 [Day (1973)]

Let X, Y be Banach spaces and let $T: X \to Y$ be bounded linear, one to -one and onto. Then the inverse T^{-1} of T is a bounded linear operator on Y.

We are ready for the inverse function theorem.

Theorem 3.3 (inverse function theorem). [Arthanasius (1973)]

Let X, Y be Banach spaces and S an open subset of X $Let\ f: S \to Y$ be C-differentiable on S. Moreover, assume that the Frechet derivative of the function is one to one and onto at some point $x, \in S.$ then the function f is locally invertible at the point $(x_0, f(x_0))$.

Proof . Let $D = f'(x_0)$. Then the operator D^{-1} exists and is defined on $Y. Moreover\ D^{-1}$ is bounded. Thus the equation $f(x) = y$ is equivalent to the equation $D^{-1}f(x) = D^{-1}y.\ fix\ y \in Y$ and define the operator $U\ on\ S$ as follows.

$$U_x = x + D^{-1}[x - f(x)], x \in S \quad \ldots \quad (3.2)$$

obviously, the fixed points of the operator X are solutions to the equation $f(x) = y$. We first determine a closed ball inside S with center at x_0 on which X is a contraction operator. To this end fix $\epsilon \in (0.1/4\ D^{-1}))\ and\ let\ \delta(\epsilon) < 0\ be\ such\ that$

$$\|[f'(x_0) - f'(x_2)](x_1 - x_2)\|\| \leq \in\| x_1 - x_2\| \quad \ldots \quad (3.3)$$

$$\|\omega\ (x_2, x_1 - x_2)\| \leq \epsilon\|x_1 - x_2\| \quad \ldots \quad (3.4)$$

For every $x_1, x_2 \in S\ with\ \|x_1 - x_0\| \leq \frac{\delta(\epsilon)}{2} \|x_2 - x_0\| \leq \delta(\epsilon)/2$

This is possible by virtue of the C- differentiability of the function f. Thus we have

$$\|Xx_1 - Xx_2\| = \|x_1 - x_2 - D^{-1}[f(x_1) - f(x_2)]\|$$
$$= \|D^{-1}f'(x_0)(x_1 - x_2)D^{-1}f'(x_2)(x_1 - x_2 - D^{-1}\omega(x_2 x_1 - x_2)\|$$
$$\leq \|D^{-1}\|\|[f'(x_0) - f'(x_2)](x_1 - x_2)\| + \|D^{-1}\|\|\omega(x_2, x_1 - x_2)\|$$
$$\leq \epsilon\|D^{-1}\|\|x_1 - x_2\| + \epsilon\|D^{-1}\|\|x_1 - x_2\|$$
$$\leq (1/2)\|x_1 - x_2\|$$

For every $x_1, x_2 \in S$ as above. It follows that X is a contraction operator on the ball $S_0(x_0)$, where $\propto = \delta(\epsilon)/2 \ \epsilon < 1/(4\|D^{-1})\|$.

Now we determine a constant $\delta > 0$ such that

$XS_o(x_0) \subset S_o(x_0) whenever\ v \in S_0(Y_0) \quad Here\ v_0 = f(x_0)$. in fact we have

$$\|Xx_0 - Xx_0\| \leq \|D^{-1}\|\|y - f(x_0)\| = \|D^{-1}\|\|y - y_o\| \leq \delta(\epsilon)/4$$

whenever

$$\|y - y_o\| \leq \delta(\epsilon)/(4\|D^{-1}\| = \beta$$

Furthermore

$$\|X - x_0\| = \|Xx - Xx_0\| + \|Xx_0 - x_0\|$$
$$\leq (1/2)\|x - x, + \delta(\epsilon)/4 \leq \delta(\epsilon)/4 + \delta(\epsilon)/4 = \delta(\epsilon)/2$$

For any $x \in S_0(x_0)$. We have shown that f is locally invertible at $(x_0, f(x_0))$, and that for any $y \in S_o(y_0)$, there exists a unique $x \in S_o(x_o) \quad With\ f(x) = y.$

Theorem 3.4 [Ince(1956)]

Let the assumption of theorem 3.3 be satisfied with the C –differentiability of the function f replaced by condition (i) of definition 3.1 then the conclusion of theorem 3.3 remains valid.

Example 3.1[Leighton (1970)] let $J = [a, b]$ and let

$$S_r = [x \in R^n : \|x\| < r]$$
$$S^r = [x \in C_n(J); \|\infty\|_\infty < r]$$

Where r is a positive number.

We consider a continuous function $F: J \text{ x } \bar{S}_o - R^n$ and the operator $X_1 S' - C_n(J)$ defined as follows:

$$(Xx)(t) = F(t, x(t)), t \in J, \infty \in S'$$

We fist note that W is continuous on S'. in fact, since, F is uniformly continuous on the compact set $J x\ \bar{S}_o \ for\ every\ \epsilon > 0\ there\ exists\ \delta(\epsilon) > 0$ such that

$\|F(t,x) - f(t,y)\| < \epsilon$

For every $x, y \in \bar{S}_o$ with $\|x - y\| < \delta(\epsilon) and\ every\ t\ J$. This implies that

$\|Ux - Uy\|_\infty < \epsilon$

Whenever $x, y \in S' with\ \|x, y\|_\infty < \delta(\epsilon)$. In order to compute the frechet derivative of X, we assume that the Jacobean matrix

$F_1(t,x) = [(\partial F/\partial x)(t,x)]\ , \qquad i = 1,2 \ldots \quad n$

exists and is continuous on $J\ x\ S,$. then given two function

$x_0 \in S'^1 (0 < r_1 < r), h \in, J\ such\ that\ x_0 + h \in C_n\ J\ x_0 + h \in S'^1$

we have

$\sup_{s\in 1} F(t + h(t) - F(t, x_0(t)) - F_1(t, x_o(t)) \| h(t) \|$

$$\leq \sup_{s\in 1}\{ \left\| [\left(\delta F/\partial x\ F(t, x_0(t))\theta, h(t)\right)] - F_1(t, x_o(t)) \right\| \|h(h)\|_\infty \qquad (3.5)$$

Where $\theta_n\ i = 1,2 \ldots n$ are function of t lying in the interval $(0,1)$. In (3.5) we have used the mean value theorem for real valued functions on S_0 as follows:

$$F_1(t, x_0(t) + h(t)) - F_1(t, \infty_0(t)) = < \nabla F_1(t_1, z_1(t) h(t) > \ i = 2 \ldots n$$

Where $z_1(t) = x_0(t) + \theta_1 h(t)$ form the uniform continuity of $\nabla F_1(t_1, z_1(t)h(t) > \ i = i \ldots n\ on\ J\ x\ S_1$ it follows that the frechet derivative $X(x_0)$ exists and is a bounded linear operator given by the formula

$$[X'(x_0)h](t) = F_1(t, x_0)(t) h(t) \qquad (3.6)$$

† Eziokwu, C. Emmanuel, Ph.D., Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria.

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