Emergence of Mathematical Foundations for Fuzziness: A Historical Perspective

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Abstract

The aim of this contribution is to present a modest historical account of the emergence of mathematical foundations for the concept of fuzziness. Several additional information and new links that have escaped the attention of earlier authors on related topics are provided.

Keywords: Fuzziness, Foundations, History

Introduction

Paraphrasing a Leibnitzian point of view, "God created this particular world of ours with the greatest plentitude and variety" (cf.[11]), and bestowed upon us some [but not "absolute"] adaptive ability to simultaneously cognize things around us, acquire knowledge based on such imprecise information, analyze and apply it to other fields of inquiry, and revise it if necessitated by logical compulsions. Thus it is cogent to believe that "shades of vagueness" must have been living with us since this world got created. Accordingly, it may not be hard to see how difficult, if not impossible, it would be to trace the exact origin of the explicit uses of certain conceptual terms that apply to all human activities, such as "vague", "hazy", and the likes.

In recent years, some papers [21,22,23] dealing with the concerned topic have appeared. This paper includes a number of additional sources that can be seen as forerunners of the founding of fuzzy mathematics. Nevertheless, no claim is made for this being a complete account.

Some history of the emergence of the concept of fuzziness

As mentioned in [21], "vagueness" is a word of modern science. Nevertheless, it seems highly unlikely that concepts involving "shades of imprecision" such as "ambiguity", "indeterminacy", "hesitancy" and the like would have completely escaped the attention of the founding fathers (Socrates, Plato, Aristotle etc.) of the "Principles of Dialogue". In particular, Plato (427-347 BC) thought of the notion of the "third region" (beyond true and false) where these opposites "tumbled about". Aristotle (384-322 BC), the founder of bi-valued logic, hinted at the possibility of a logical system with more than two truth values. Moreover, such ideas might have percolated into the prolific writings of well-known logicians and philosophers of the medieval period until the 15th century or so. It is my presumption that the dominance of Aristotelian logic and Euclidean mathematics would have over-shadowed efforts to undertake a rigorous treatment of imprecisely defined classes of objects of the real physical world for a considerably long period of time.

As mentioned in [21], John Locke (1632-1704) used the word "vague" in his "Essay on human understanding" (1689). F.W. Nietzsche (1844-1900), who deliberated on imprecisely defined concepts, rejected the notion of "objective reality" and claimed that "knowledge is contingent and conditional" and related to "fluid perspectives" [Wikipedia: "Perspectivism"].

To my knowledge, no substantial contribution on the concept of vagueness is known until Gottlob Frege's (1844-1925) "Basic Laws of Arithmetic" [5], originally published in German (2 vols. 1893-1906), appeared. Frege is known to be the first to use the word "vague" as a technical term. It may be noted that the most distinctive idea defining "fuzzy objects" as "objects with unsharp boundaries" [28] has a counterpart in Frege [5] where "crisp objects" are defined as "objects with sharp boundaries". Frege did not envisage the possibility of a mathematical theory of vague concepts. On the contrary, he emphasized that a vague concept cannot be considered as a concept.

The roots of "fuzziness" can be traced in Richard Dedekind's (1831-1916) remark [4]: "… In this way, we reach a notion, very useful in many cases, of systems [sets] in which every element is endowed with a certain frequency number which indicates how often it is to be reckoned as an element of the system … ". It may be noted that Menger [13] mentioned that if the "frequency number" is suitably interpreted (say, if [0,1] be the range of membership values) as the degree of membership of an element of a system, we arrive at the notion of a "fuzzy system" (see the observations provided in section 3).

Bertrand Russell (1872-1970) provided a detailed description of vagueness [20]. He defined the concept of vagueness "as a matter of degree" or "as the existence of fluent boundaries" or "as a conception applicable to every kind of representation". Russell considered a representation as "vague" if the relation of the representing system to the represented one is "one-many" and as "accurate" if the said relation is "one-one". For example, "a smudged photograph is vague", "a small-scale map is vaguer than a large-scale map", etc. He emphasized that "vagueness" is distinct from "generality". According to him, vagueness applies to all that is cognitive whereas a proposition involving generality can be verified by a number of facts. For example, “This is a man” is verifiable if man were a precise idea.

C.S. Peirce (1883-1914) deliberated on the concept of vagueness and concluded that “a proposition is vague when there are possible states of things concerning which it is intrinsically uncertain whether, had they been contemplated by the speaker, he would have regarded them as excluded or allowed by the proposition” ([16,17], cf. [21]).

Max Black (1909-1988) defined vagueness in terms of “borderline cases” i.e. “… a state of impossibility to decide either way” [2,3] which can be considered as the kernel of Russell’s and Pierce’s approaches. He further elaborated: “the vagueness of a term could be usually indicated by producing some statement that situations are conceivable in which nobody would know to use it or in which it would be impossible either to assert or deny its application”. In particular, he argued that vagueness is distinct from “ambiguity”, “generality”, and “indeterminacy”. Black emphasized that a word is “general” if the field of its application is finite and specified by a one-many relation whereas it is “vague” if the field of its application is finite and devoid of any specification of its boundary. He treated “indeterminacy” as a characteristic of vagueness. A term is vague if its application leads to indeterminate cases. Indeterminacy, in contrast to ambiguity, can be characterized by a situation which allows both “affirmation and denial”. Ambiguity arises when the choices between two or more alternatives are unspecified. Black [2] pointed out that Russell’s definition of vagueness as constituted by a “one-many relation …” is held to confuse vagueness with generality. An elaborate description of the notions of ambiguity, generality, indeterminacy, and vagueness can be found in some recent papers [6,29].

Tadeusz Kortabinski (1886-1981) seems to be the first to use the word “grades” (chwiejue, in polish) in defining vagueness. For him, the concept of vagueness involved a “continuum of grades” (cf. [21]).

Kazimierz Adjukiewicz (1890-1963) stated that “a term is vague if and only if its use in a context decidable will make the context undecidable …” (cf. [21]).

Ludwik Fleck (1896-1961), a medical doctor, dwelt upon the elements of “indeterminacy” inherently associated with problems encountered in determining correspondences between the class of “symptoms” and that of "diseases" (cf. [2,21]) i.e. "in quantifying clinical (diagnostic and prognostic) judgements" [25]. See [22] for many technical details discussed in this regard.

Herman Weyl (1885-1955) defined the relation of equality in the physical continuum: "… the individuals in the same complete state (no further refinement is possible) are indiscernible by any intrinsic characters although they may not be the same thing" [24]. The tripartite relation viz. "objects may be identical, distinct or twins" was exploited by Parker-Rhodes (1914-1987) to develop his theory of "sorts" [15] which he utilised to study some fundamental problems of physics.

Karl Menger (1902-1985) in his study of a number of mathematical concepts applied probabilistic methods, a well-established tool by then for solving stochastically defined problems in mathematics, statistics, physics, and other sciences. Menger, intrigued by Henri Poincare's radical advocation that the relation of equality (=) in the "physical continuum" is "non-transitive" (A=B, B=C, A≠C), advanced a probabilistic interpretation of the equality of elements in the physical continuum. He suggested that the equality of two objects A and B of the physical continuum could be described by "associating with A and B a number, namely, the probability of finding A and B indistinguishable" while in applications, this number would be represented by the "relative frequency of the cases in which A and B are not distinguished". Menger emphasized that, unlike mathematical continuum, objects of the physical continuum may be indistinguishable but not identical. In course of developing his microgeometry to study the complex structure of the observable physical continuum, Menger introduced [12,13] a "tripartite" relation viz., "objects may be indistinguishable, apart or overlapping", which was a radical departure from classical mathematics. In [12], he introduced the notion of "ensemble flous" (a.k.a hazy sets) in which the classical elementhood relation was replaced by "probabilistic relation", and pointed out that these sets are not, in general, equivalence classes as the relation of indistinguishability on various probabilistic levels is not transitive. Based on the overlapping characteristic of such sets, Menger remarked ("overly", I think, see the justification provided in section 3) that his "hazy sets" could be seen as Zadeh's "fuzzy sets" if the "probability of an element belonging to a set" is taken as the "degree of an …" [13].

Jan Lukasiewicz (1878-1956) was the first to systematically develop many-valued logic (precisely n-valued logic, n>2) as a generalization of the classical two-valued logic in the 1920s (details can be found in [18], for example).

Notwithstanding, it was C.S. Peirce who attempted first to formulate a many-valued logic (cf. [19]). During (1932-1939), another infinite-valued logic, known as G_{∞} , was developed by Kurt Godel (cf. [7]).

1. Emergence of fuzzy set theory and fuzzy logic

As described in section 2, by the early 20th century it became more perceptive to see that every proposition that can be found in practice has a certain degree of vagueness and also that such propositions would require a continuum of truth values for their effective characterization. In particular, it became widely recognized that most human decision making would remain characteristically imprecise. In turn, managing a multitude of information, inherently loaded with uncertainties, became the focus of our inquiries.

The emergence of computers in the mid-20th century, besides being an important tool for information processing, raised a number of challenges as well. In particular, for automating human decisions, usually expressed in terms of imprecise words of a natural language, it became a pressing need to develop a suitable framework for translating natural language expressions into a machine language.

Above all, it was a great insight of Lotfi Zadeh [1921-2017] that made him recognize that dealing effectively with systems (e.g. biological systems) more complex than man-made systems would require, instead of conventional mathematics, the mathematics of fuzzy or cloudy quantities which are not described in terms of probability distributions [28]. Zadeh was the first to use the word "fuzzy" as a technical term [1]. He emphasized that the notion of fuzziness is fundamentally distinct from that of vagueness, the former arises due to the absence of "sharp referential boundary" whereas the latter ensues due to the presence of "one-to-many relationship between the representing system and the represented one". Zadeh founded "fuzzy set theory" and correspondingly "fuzzy logic" as an appropriate mathematical framework to deal with classes possessing unsharp boundaries.

A "fuzzy set" ("class") A in a universe set X is characterized by a "membership function" usually denoted by $\mu_A(x)$ or $f_A(x)$, which associates with each point in X a real number in the interval [0,1] or in a suitable partially ordered set. The value of $\mu_A(x)$ at x represents the "grade or degree of membership" of x in A. Formally, a fuzzy set A in X is a mapping μ_A: X

→[0,1]. See [7,8,9,27] for a detailed study of fuzzy set theory. The applications of fuzzy sets abound, particularly in Pattern Classification and Information Processing. In view of having a wide potential for applicability of fuzzy set theory, its several extensions and generalizations have kept on appearing, and several new challenges are yet to come to the fore (see [1,9,10,26,27] for details).

Fuzzy logic, also sometimes called "diffuse logic" [18], is a many-valued logic with a continuum of degrees of truth value of a proposition in the interval [0,1]. Essentially, as classical logic corresponds to crisp sets, so does fuzzy logic to fuzzy sets. It may be re-iterated that, prior to the emergence of fuzzy logic, two well-known infinite-valued logics (Godel logic $G_\infty (1932 - 1939)$ and Lukasiewicz logic L_∞ (1930), both with truth degree set [0,1], have been present [7]. Fuzzy logic has found enormous applications both in software and hardware, particularly in system theory: Expert systems, information retrieval systems, control systems etc. In all these, it is fuzzy logic that provides means for translating fuzzy knowledge into precise formulas (see [9,14,19] for details).

Zadeh emphasized that the fuzzy set theory is designed to provide a suitable framework to deal with problems in which the source of imprecision is the absence of sharply defined criteria of class membership, rather than the presence of random variables. He pointed out that the notion of a fuzzy set is completely non-statistical in nature [27,28]. Indeed, in order to deal with fuzzy knowledge (expert's knowledge expressed in terms of imprecise words of a natural language), probabilistic approaches are not appropriate. Probabilistic approaches can deal with problems in which uncertainty arises due to statistical variation, and these are couched in the Bayesian framework. Essentially "frequency" interpretation of uncertainty underlies probabilistic methods whereas "possibility" interpretation underlies fuzziness. The membership function of a fuzzy set is characterized by "subjective measures", not by "probability functions". The membership grades are not probabilities (see [8,10,14,19], for details).

Akin to the point of view that classical logic and set theory constitute a foundation for classical mathematics, fuzzy logic and, fuzzy set theory constitute a foundation for fuzzy mathematics. It may be noted that to lay the foundation for fuzzy set theory , "set-theoretic" approaches, "model-theoretic" approaches, and "category-theoretic" approaches have been provided (see [7,23], for a historical, comparative and exhaustive treatment of this topic).

2. Concluding Remarks

In this paper, a comprehensive historical perspective of the emergence of mathematical foundations for the concept of fuzziness has been provided. All the items included in the list of references are duly acknowledged. At this end, I would wish to sincerely apologize to all those people whose writings on this topic might have been inadvertently missed out.

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